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OFF-SHELL CLOSED STRING AMPLITUDES: TOWARDS A COMPUTATION OF THE TACHYON POTENTIAL

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ABSTRACT

We derive an explicit formula for the evaluation of the classical closed string action for any off-shell string field, and for the calculation of arbitrary off-shell amplitudes. The formulae require a parametrization, in terms of some moduli space coordinates, of the family of local coordinates needed to insert the off-shell states on Riemann surfaces. We discuss in detail the evaluation of the tachyon potential as a power series in the tachyon field. The expansion coefficients in this series are shown to be geometrical invariants of Strebel quadratic differentials whose variational properties imply that closed string polyhedra, among all possible choices of string vertices, yield a tachyon potential which is as small as possible order by order in the string coupling constant. Our discussion emphasizes the geometrical meaning of off-shell amplitudes.

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1. Introduction and Summary

A manifestly background independent field theory of strings should define the conceptual framework for string theory and should allow the precise definition and explicit computation of nonperturbative effects. The present version of quantum closed string field theory, developed explicitly only for the case of bosonic strings, while not manifestly background independent, was proven to be background independent for the case of nearby backgrounds [1,2,3]. The proof indeed uncovered structures that are expected to be relevant to the conceptual foundation of string theory. At the computational level one can ask if present day string field theory allows one to do new computations, in particular computations that are not defined in first quantization. While efficient computation may require the manifestly background independent formulation not yet available, it is of interest to attempt new computations with present day tools. This is the main purpose of the present paper.

Off-shell amplitudes are not naturally defined without a field theory. Indeed, while the basic definition of an off-shell string amplitude *is* given in first quantization, off-shell string amplitudes are only interesting if they obey additional properties such as permutation symmetry and consistent factorization. These properties are automatically incorporated when the off-shell amplitudes arise from a covariant string field theory [4].

Off-shell string amplitudes are obtained by integrating over the relevant moduli space of Riemann surfaces differential forms that correspond to the correlators of vertex operators inserted at the punctures of the surfaces and antighost line integrals. The vertex operators correspond to non-primary fields of the conformal field theory. In contrast, in on-shell string amplitudes the vertex operators are always primary fields. In order to insert non-primary fields in a punctured Riemann surface we must choose an analytic local coordinate at every puncture. The moduli space of Riemann surfaces of genus g and N punctures is denoted as $\overline{\mathcal{M}}_{g,N}$, and the moduli space of such surfaces with choices of local coordinates at the punctures is denoted as $\widehat{\mathcal{P}}_{g,N}$.[‡] An off-shell amplitude is just an integral over a subspace of $\widehat{\mathcal{P}}_{g,N}$. Typically, the relevant subspaces of $\widehat{\mathcal{P}}_{g,N}$ are sections over $\overline{\mathcal{M}}_{g,N}$. Such sections are obtained by making a choice of local coordinates at every puncture of each surface in $\overline{\mathcal{M}}_{g,N}$. In closed string field theory, the use of minimal area metrics allows one to construct these sections using the vertices of the theory and the propagator. Off-shell amplitudes arising in open string field theory have been studied by Samuel [5,6].

While interesting in their own right, off-shell amplitudes are not physical observables. More relevant is the evaluation of the string action for any choice of an off-shell string field. This computation would be necessary in evaluating string instanton effects. The string action, apart for the kinetic term, is the sum of string interactions each of which is defined by a *string vertex*, namely, a subspace $\mathcal{V}_{g,N}$ of $\widehat{\mathcal{P}}_{g,N}$. Typically $\mathcal{V}_{g,N}$ is a section over a compact subspace of $\overline{\mathcal{M}}_{g,N}$.

‡ The local coordinate at each puncture is defined only up to a constant phase.

Therefore, given an off-shell string field, the contribution to the string action arising from a particular interaction corresponds to a *partially integrated* off-shell amplitude. The classical potential of a field theory in flat Minkowski space is a simple example of the above considerations; it amounts to the evaluation of the action for field configurations that are spacetime constants. Ideally we would like to compute, for the case of bosonic strings formulated around the twenty-six dimensional Minkowski space, the complete classical potential for the string field. This may be eventually possible but we address here the computation of the classical potential for some string modes. In particular we focus in the case of the tachyon of the closed bosonic string.

For the case of open strings some interesting results have been obtained concerning the classical *effective* potential for the tachyon [7]. This potential takes into account the effect of all other fields at the classical level. In the context of closed string field theory only the cubic term in the tachyon potential is known [8]. The possible effects of this term have been considered in Refs.[9,10,11]. Our interest in the computation of the closed string tachyon potential was stimulated by G. Moore [12] who derived the following formula for the potential $V(\tau)$ for the tachyon field $\tau(x)$

$$V(\tau) = -\tau^2 - \sum_{n=3}^{\infty} \mathrm{v}_{_N} \frac{\tau^N}{N!}\,, \qquad \text{where} \quad \mathrm{v}_{_N} \sim \int\limits_{\mathcal{V}_{0,N}} \Big(\prod_{I=1}^{N-3} \mathrm{d}^2 h_{_I}\Big) \prod_{I=1}^{N} \left|h'_{_I}(0)\right|^{-2} . \tag{1.1}$$

This potential is the tachyon potential with all other fields set to zero. It is not an effective potential. It is fully nonpolynomial, and starts with a negative sign quadratic term (the symbols appearing in the expression for $\mathrm{v}_{_N}$ will be defined in sect.2). The calculation of the tachyon potential amounts to the calculation of the constant coefficients $\mathrm{v}_{_N}$ for $N \geq 3$. For the cubic term, since $\mathcal{V}_{0,3}$ is a point, the integral is actually not there, and the evaluation of the coefficient of v_3 is relatively straightforward. The higher coefficients are difficult to compute since they involve integrals over the pieces of moduli spaces $\mathcal{V}_{0,N}$.

We will rewrite (1.1) in PSL$(2,\mathbb{C})$ invariant form in order to understand the geometrical significance of the coefficients $\mathrm{v}_{_N}$ and to set up a convenient computation scheme. Moreover, we will obtain a generalization of eqn.(1.1) valid for any component field of the string field theory. The expression will be given in the operator formalism and will be PSL$(2,\mathbb{C})$ invariant.

<u>Extremal Property.</u> We will show that the polyhedral vertices of closed string field theory are the solution to the problem of minimizing recursively the coefficients in the expansion defining the tachyon potential. That is, the choice of the Witten vertex, among all possible choices of cubic string vertices[§] minimizes $\mathrm{v}_{_3}$. Once the three string vertex is chosen, the region of moduli space to be covered by the four string vertex is fixed. The choice of the standard tetrahedron

§ Vertices are defined by coordinate curves surrounding the punctures and defining disks. The disks should not have finite intersection

for the four string vertex, among all possible choices of four string vertex filling the required region, will minimize the value of v_4. This continues to be the case for the complete series defining the classical closed string field theory. This fact strikes to us as the string field theory doing its best to obtain a convergent series for the tachyon potential. It is also interesting that a simple demand, that of minimizing recursively the coefficients of the tachyon potential, leads uniquely to the polyhedral vertices of classical closed string field theory. It has been clear that the consistency of closed string field theory simply depends on having a choice of string vertices giving a cover of moduli space. The off-shell behavior, however, is completely dependent on the choice of vertices, and one intuitively feels there are choices that are better than other. We see here nice off-shell behavior arising from polyhedra.

A Minimum in the Potential? In calculating the tachyon potential we must be very careful about sign factors. The relative signs of the expansion coefficients are essential to the behavior of the series. We find that all the even terms in the tachyon potential, including the quadratic term, come with a negative sign, and all the odd terms come with a positive sign. It then follows, by a simple sign change in the definition of the tachyon field, that all the terms in the potential have negative coefficients. This implies that there is no global minimum in the potential since the potential is not bounded from below. Moreover, there is no local minimum that can be identified without detailed knowledge of the complete series defining the tachyon potential. If the series defining the potential has no suitable radius of convergence further complications arise in attempting to extract physical conclusions. We were not able to settle the issue of convergence, but present some work that goes in this direction. In estimating the coefficient v_N we must perform an integral of the tachyon off-shell amplitude over $\mathcal{V}_{0,N}$. In this region the tachyon amplitude varies strongly. In the middle region the amplitude is lowest, and if this were the dominating region, we would get convergence. In some corners of $\mathcal{V}_{0,N}$ the amplitude is so big that, if those corners dominate, there would be no radius of convergence.

It is important to emphasize that only the tachyon effective potential (or the full string field potential) is a significant object. The tachyon potential is not by itself sufficient to make physical statements. A stable critical point of this potential may not even be a critical point of the complete string field potential. The effects of the infinite number of massive scalar fields must be taken into account. Our results, making unlikely the existence of a stable critical point reinforce the sigma-model arguments that suggest that bosonic strings do not have time independent stable vacua [13], but are not conclusive. (See also Ref.[14] for a discussion of tachyonic ambiguities in the sigma model approach to the string effective action.) The calculation of the full string field potential, or the tachyon effective potential is clearly desirable. We discuss in Appendix A string field redefinitions, and argue that it does not seem possible to bring the string action into a form (such as one having a purely quadratic tachyon potential) where one can easily rule out the existence of a local minimum.

General Off-Shell Amplitudes. Since general off-shell computations do not have some of the simplifying circumstances that are present for the tachyon (such as being primary, even off-

shell), we derive a general formula useful to compute arbitrary off-shell amplitudes. This formula, written in the operator formalism, gives the integrand for generic string amplitudes as a differential form in $\widehat{\mathcal{P}}_{0,N}$. The only delicate point here is the construction of the antighost insertions for Schiffer variations representing arbitrary families of local coordinates (local coordinates at the punctures as a function of the position of the punctures on the sphere). Particular cases of this formula have appeared in the literature. If the family of local coordinates happens to arise from a metric, the required antighost insertions were given in Ref.[15]. Antighost insertions necessary for zero-momentum dilaton insertions were calculated in Refs.[16].

<u>Organization of the Contents of this Paper.</u> We now give a brief summary of the contents of the present paper. In sect.2 we explain what needs to be calculated to extract the tachyon potential, set up our conventions, and summarize all our results on the tachyon potential. In sect.3 we prepare the grounds for the geometrical understanding of the off-shell amplitudes. We review the definition of the mapping radius of punctured disks and study its behavior under PSL(2, $\mathbb{C}$) transformations (the conformal maps of the Riemann sphere to itself). We show how to construct PSL(2, $\mathbb{C}$) invariants for punctured spheres equipped with coordinate disks, by using the mapping radii of the punctured disks and coordinate differences between punctures, both computed using an arbitrary uniformizer. We review the extremal properties of Jenkins-Strebel quadratic differentials [17], and show how our PSL(2, $\mathbb{C}$) invariants, in addition to having extremal properties, provide interesting (and seemingly new) functions on the moduli spaces $\overline{\mathcal{M}}_{0,N}$.⋆ In sect.4 we compute the off-shell amplitude for scattering of N tachyons at arbitrary momentum, and give the answer in terms of integrals of PSL(2, $\mathbb{C}$) invariants. This formula is the off-shell extension of the Koba-Nielsen formula. At zero momentum and partially integrated over moduli space, it gives us, for each N, the coefficient v_{N} of the tachyon potential. We show why these coefficients are minimized recursively by the string vertices defined by Strebel differentials. In sect.5 we do large$-N$ estimates for the coefficients v_{N} of the tachyon potential in an attempt to establish the existence of a radius of convergence for the series. The measure of integration is computed exactly for corners in $\mathcal{V}_{0,N}$ representing a planar configuration for the tachyon punctures on the sphere. We are also able to estimate the measure of integration for a uniform distribution of punctures on the sphere. In sect.6 we give the operator formalism construction for general differential forms on $\widehat{\mathcal{P}}_{0,N}$ labelled by arbitrary off-shell states.

⋆ These functions are analogous to the function that assigns to an unpunctured Riemann surface the area of the minimal area metric on that surface.

2. String Action and the Tachyon Potential

In this section we will show what must be calculated in order to obtain the tachyon potential. This will help put in perspective the work that will be done in the next few sections. We will also give some of the necessary conventions, and we will comment on the significance of the tachyon potential and the limitations of our results. All of our results concerning the tachyon potential will be summarized here.

The full string field action is a non-polynomial functional of the infinite number of fields, and from the component viewpoint, a non-polynomial function of an infinite number of space-time fields. Here we consider only the part of it which contains the tachyon field $\tau(x)$. We will call this part the tachyonic action $S^{\rm tach}(\tau)$. It is a nonpolynomial, non-local functional of the tachyonic field $\tau(x)$. In order to introduce the string field configuration associated to the tachyon field $\tau(x)$ we first Fourier transform

$$\tau(p) = \int \mathrm{d}^D x \;\tau(x)\, e^{-ipx}\,, \tag{2.1}$$

and use $\tau(p)$ to define the tachyon string field $|T\rangle$ as follows

$$|T\rangle = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\,\tau(p)\, c_1\bar{c}_1|\mathbf{1},p\rangle. \tag{2.2}$$

In the conformal field theory representing the bosonic string, the tachyon vertex operator is given by $T_p = c\bar{c}e^{ipX}$ and is of conformal dimension $(L_0,\overline{L}_0) = (-1+p^2/2, -1+p^2/2)$. The conformal field theory state associated to this field is $T_p(0)|\mathbf{1}\rangle = c_1\bar{c}_1|\mathbf{1},p\rangle$. This state is BRST invariant when we satisfy the on-shell condition $L_0 = \overline{L}_0 = 0$, which requires $p^2 = 2 = -M^2$ (this is the problematic negative mass squared of the tachyon). The above representative T_p for the cohomology class of the physical tachyon is particularly nice, because this tachyon operator remains a primary field even off-shell ($p^2 \neq 2$).

The tachyonic action is then given by evaluating the string field action $S(|\Psi\rangle)$ for $|\Psi\rangle = |T\rangle$:

$$S^{\rm tach}(\tau) = S\;(\,|\Psi\rangle = |T\rangle\,)\,, \tag{2.3}$$

where

$$S(\Psi) = \frac{1}{2}\langle\Psi|c_0^- Q|\Psi\rangle + \sum_{N=3}^{\infty}\frac{\kappa^{N-2}}{N!}\{\Psi^N\}_{\mathcal{V}_{0,N}}, \tag{2.4}$$

and κ is the closed string field coupling constant (see [18]). This action satisfies the classical master equation $\{S,S\} = 0$ when the string vertices $\mathcal{V}_0 = \sum_{N\geq 3}\mathcal{V}_{0,N}$ are chosen to satisfy the recursion relations $\partial\mathcal{V}_0 + \frac{1}{2}\{\mathcal{V}_0,\mathcal{V}_0\} = 0$ (see Ref.[3])

Let us verify that the above definitions lead to the correctly normalized tachyon kinetic term

$$S^{\rm tach}_{\rm kin} = \frac{1}{2}\,\langle T|c_0^- Q|T\rangle\,. \tag{2.5}$$

Recall that the BRST operator Q is of the form $Q = c_0 L_0 + \bar{c}_0\bar{L}_0 + \cdots$, where the dots denote the terms which annihilate $|T\rangle$. Moreover, acting on the state $c_1\bar{c}_1|\mathbf{1},p\rangle$ the operators L_0 and $\bar{L}_0$ both have eigenvalue $p^2/2-1$. We then find

$$S^{\rm tach}_{\rm kin} = \frac{1}{2}\int\frac{\mathrm{d}^D p}{(2\pi)^D}\int\frac{\mathrm{d}^D p'}{(2\pi)^D}\langle -p',\mathbf{1}|c_{-1}\bar{c}_{-1}c_0^- c_0^+ c_1\bar{c}_1|p,\mathbf{1}\rangle\;\tau(p')\,(p^2-2)\,\tau(p). \tag{2.6}$$

We follow the conventions of Ref.[18] where

$$\langle -p',\mathbf{1}|c_{-1}\bar{c}_{-1}c_0^- c_0^+ c_1\bar{c}_1|p,\mathbf{1}\rangle \equiv (2\pi)^D\,\langle -p',\mathbf{1}^c|p,\mathbf{1}\rangle = (2\pi)^D\delta^D(p'+p)\,, \tag{2.7}$$

and $c_0^{\pm} = \frac{1}{2}(c_0 \pm \bar{c}_0)$. Using this we finally find

$$S^{\rm tach}_{\rm kin} = -\frac{1}{2}\int\frac{\mathrm{d}^D p}{(2\pi)^D}\tau(-p)(p^2-2)\tau(p)\,, \tag{2.8}$$

which is indeed the correctly normalized kinetic term.* The N-th term in the expansion of the tachyonic action requires the evaluation of string multilinear functions

$$S^{\rm tach}_{0,N}(\tau) = \frac{\kappa^{N-2}}{N!}\{T^N\}_{\mathcal{V}_{0,N}}\,, \tag{2.9}$$

and this will be one of the main endeavors in this paper. The answer will be of the form

$$\{T^N\}_{\mathcal{V}_{0,N}} = \int\prod_I\frac{\mathrm{d}p_I}{(2\pi)^D}\;(2\pi)^D\delta\left(\sum p_I\right)\cdot V_N(p_1,\cdots p_N)\,\tau(p_1)\cdots\tau(p_N)\,, \tag{2.10}$$

where V, the function we will be calculating, is well defined up to terms that vanish upon use of momentum conservation. To extract from this the tachyon potential we evaluate the above term in the action for spacetime constant tachyons $\tau(x)=\tau_0$, which gives $\tau(p) = \tau_0(2\pi)^D\delta(p)$,

$\star$ We work in euclidean space with positive signature, and the action S should be inserted in the path integral as $\exp(S/\hbar)$, which is a convenient convention in string field theory. The euclidean action S is of the form $S = -\int \mathrm{d}^D x(K+V)$, where K and V stand for kinetic and potential terms respectively.

and as a consequence

$$S^{\rm tach}_{0,N}(\tau_0) = \frac{\kappa^{N-2}}{N!} V_N(\mathbf{0})\, \tau_0^N \cdot (2\pi)^D \delta(\mathbf{0})\,. \tag{2.11}$$

Since the infinite $(2\pi)^D\delta(\mathbf{0})$ factor just corresponds to the spacetime volume, the tachyon potential will read

$$V(\tau) = -\tau^2 - \sum_{N\geq 3} \frac{\kappa^{N-2}}{N!}\, \mathrm{v}_{_N}\ \tau^N\,, \tag{2.12}$$

where we have used the fact that the potential appears in the action with a minus sign. Here the expansion coefficients $\mathrm{v}_{_N}$ are given by

$$\mathrm{v}_{_N} \equiv V_{_N}(\mathbf{0})\,. \tag{2.13}$$

We will see that the coefficient $\mathrm{v}_{_3}$ is given by†

$$\mathrm{v}_3 = -\frac{3^9}{2^{11}} \approx -9.61\,. \tag{2.14}$$

Analytic work, together with numerical evaluation gives [19]

$$\mathrm{v}_{_4} = 72.39\ \pm 0.01\,. \tag{2.15}$$

Therefore, to this order the tachyon potential reads

$$V(\tau) = -\tau^2 + 1.602\kappa\tau^3 - 3.016\kappa^2\tau^4 + \cdots\ , \tag{2.16}$$

and gives no local minimum for the tachyon. The general form for $\mathrm{v}_{_N}$ will be shown to be given by

$$\mathrm{v}_{_N} = (-)^N \frac{2}{\pi^{N-3}} \int_{\mathcal{V}_{0,N}} \prod_{I=1}^{N-3} \frac{\mathrm{d}x_{_I} dy_{_I}}{\rho^2_{_I}}\ \frac{1}{\rho^2_{_{N-2}}(0)\rho^2_{_{N-1}}(1)\rho^2_{_N}(\infty)}\,, \tag{2.17}$$

where the quantities $\rho_{_I}$, called mapping radii, will be discussed in the next section. Since the integrand is manifestly positive, v_N will be positive for even N and negative for odd N. Note that by a sign redefinition of the tachyon field we can make all terms in the tachyon potential negative. Therefore the tachyon potential is unbounded from below and cannot have a global minimum. A local minimum may or may not exist. Even these statements should be qualified if the series defining the tachyon potential has no suitable radius of convergence. We will study the large-N behavior of the coefficients $\mathrm{v}_{_N}$ in sect.5, but we will not be able to reach a definite conclusion as far as the radius of convergence goes.

† The value quoted here agrees with that quoted in Ref.[9] after adjusting for a factor of two difference in the definition of the dimensionless coupling constant.

Even if one could establish the existence of a local minimum for the tachyon potential, the question remains whether it represents a vacuum for the whole string field theory. One way to address this question would be to compute the effective potential for the tachyon. For a complete understanding of the string field potential we should actually examine all zero-momentum Lorentz scalar fields appearing in the theory. This would include physical scalars, unphysical scalars and trivial scalars. Since even the number of physical scalars at each mass level grows spectacularly fast [20], a more stringy way to discuss the string field potential is clearly desirable.

3. Geometrical Preliminaries

In the present section we will begin by reviewing the definition of mapping radius of a punctured disk. While this object requires a choice of local coordinate at the puncture, it is possible to use it to construct conformal invariants of spheres with punctured disks *without* having to make choices of local coordinates at the punctures. We will discuss in detail those invariants. We review the extremal properties of the Strebel quadratic differentials and explain how to calculate mapping radii from them. The invariants relevant to the computation of tachyon amplitudes are shown to have extremal properties as well.

3.1. Reduced Modulus and PSL(2,$\mathbb{C}$) Invariants

Given a punctured disk D, equipped with a chosen local coordinate z vanishing at the puncture, one can define a conformal invariant called the mapping radius $\rho_{_D}$ of the disk. It is calculated by mapping conformally the disk D to the unit disk $|w| \leq 1$, with the puncture going to $w = 0$. One then defines

$$\rho_{_D} \equiv \left| \frac{dz}{dw} \right|_{w=0} . \tag{3.1}$$

Alternatively one may map D to a round disk $|\xi| \leq \rho_{_D}$, with the puncture going to $\xi = 0$, so that $|dz/d\xi|_0 = 1$. The reduced modulus M_D of the disk D is defined to be

$$M_D \equiv \frac{1}{2\pi} \ln \rho_{_D} \ . \tag{3.2}$$

Clearly, both the mapping radius and the reduced modulus depend on the chosen coordinate. If we change the local coordinate from z to z', also vanishing at the puncture, we see using (3.1) that the new mapping radius $\rho'_{_D}$ is given by

$$\rho'_{_D} = \rho_{_D} \left| \frac{dz'}{dz} \right|_{z=0}, \qquad \rightarrow \qquad \frac{\rho_{_D}}{|dz|} = \text{invariant}. \tag{3.3}$$

Thus the mapping radius transforms like the inverse of a conformal metric g, for which the

length element $g|dz|$ is invariant. For the reduced modulus we have

$$M'_D = M_D + \frac{1}{2\pi} \ln \left| \frac{dz'}{dz} \right|_{z=0} . \tag{3.4}$$

<u>PSL(2, $\mathbb{C}$) Invariants</u> It should be noted that the above transformation property (3.3) is not in contradiction with the conformal invariance of the mapping radius. Conformal invariance just states that if we map a disk, *and* carry along the chosen local coordinate at the puncture, the mapping radius does not change. This brings us to a point that will be quite important. Throughout this paper we will be dealing with punctured disks on the Riemann sphere. How will we choose local coordinates at the punctures? It will be done as follows: we will choose a global uniformizer z on the sphere and keep it fixed. If a punctured disk D has its puncture at $z = z_0$, then the local coordinate at the puncture will be taken to be $(z - z_0)$ (the case when $z_0 = \infty$ will be discussed later). Consider now an arbitrary PSL(2, $\mathbb{C}$) map taking the sphere into itself

$$z \to f(z) = \frac{az + b}{cz + d} , \tag{3.5}$$

Under this map a disk D centered at $z = z_0$ will be taken to a disk $f(D)$ centered at $z = f(z_0)$. According to our conventions, the local coordinate for D is $z - z_0$ and the local coordinate for $f(D)$ is $z - f(z_0)$. This latter coordinate is not the image of the original local coordinate under the map. Therefore the mapping radius *will transform*, and we can use (3.3) to find

$$\rho_{f(D)} = \rho_D \left| \frac{df}{dz} \right|_{z_0} , \qquad \to \qquad \rho_D = |cz_0 + d\,|^2 \, \rho_{f(D)} \, . \tag{3.6}$$

The off-shell amplitudes will involve the mapping radii of various disks. Moreover, they must be PSL(2, $\mathbb{C}$) invariant. How can that be achieved given that we do not have a PSL(2, $\mathbb{C}$) invariant definition of the mapping radius? Let us first examine the case when we have two punctured disks on a sphere. The data is simply a sphere with two marked points and two closed Jordan curves each surrounding one of the points. We will associate a PSL(2, $\mathbb{C}$) invariant to this sphere. The invariant is calculated using a uniformizer, but is independent of this choice. Choose any uniformizer z on the sphere, and denote the disks by $D_1(z_1)$ and $D_2(z_2)$, where z_1 and z_2 are the positions of the punctures. We now claim that

$$\chi_{12} \equiv \frac{|z_1 - z_2|^2}{\rho_{D_1(z_1)} \, \rho_{D_2(z_2)}} , \tag{3.7}$$

is a PSL(2, $\mathbb{C}$) invariant (in other words, it is independent of the uniformizer, or, it is a conformal invariant of the sphere with two punctured disks). Indeed, under the PSL(2, $\mathbb{C}$) transformation

given in (3.5) we have that

$$|z_1 - z_2| = |cz_1 + d\,| \cdot \, |cz_2 + d\,| \cdot \, |f(z_1) - f(z_2)|\,, \tag{3.8}$$

and it follows immediately from this equation and (3.6) that

$$\frac{|z_1 - z_2|^2}{\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}} = \frac{|f(z_1) - f(z_2)|^2}{\rho_{f(D_1(z_1))}\,\rho_{f(D_2(z_2))}}\,, \tag{3.9}$$

which verifies the claim of invariance of the object χ_{12}. It seems plausible that any PSL$(2,\mathbb{C})$ invariant built from mapping radii of two disks must be a function of χ_{12}. It is not hard to construct in the same fashion an PSL$(2,\mathbb{C})$ invariant of three punctured disks. Indeed, we have

$$\chi_{123} \equiv \frac{|z_1 - z_2|\,|z_1 - z_3|\,|z_2 - z_3|}{\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}\rho_{D_3(z_3)}}\,, \tag{3.10}$$

which is easily verified to be a conformal invariant. This invariant can be written in terms of the invariant associated to two disks, one sees that

$$\chi_{123} = (\,\chi_{12}\,\chi_{13}\,\chi_{23}\,)^{1/2}\,. \tag{3.11}$$

This shows the invariant χ_{123} of three disks is not really new. It is also clear that we can now construct many invariants of three disks. We can form linear combinations of complicated functions build using the invariants associated to all possible choices of two disks from the three available ones. Nevertheless, the particular invariant χ_{123} given above will be of relevance to us later on. Let us finally consider briefly the case of four punctured disks, and concentrate on invariants having a product of all mapping radii in the denominator. Let

$$\chi_{1234} \equiv \frac{|z_1 - z_2|\,|z_2 - z_3|\,|z_3 - z_4||z_4 - z_1|}{\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}\rho_{D_3(z_3)}\rho_{D_4(z_4)}}\,, \tag{3.12}$$

and, as the reader will have noticed, the only requisite for invariance is that, as it happens above, every z_I appear twice in the numerator. This can be done in many different ways; for example, we could have written

$$\chi'_{1234} \equiv \frac{|z_1 - z_2|^2\,|z_3 - z_4|^2}{\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}\rho_{D_3(z_3)}\rho_{D_4(z_4)}}\,, \tag{3.13}$$

and the ratio of the two invariants is

$$\frac{\chi'_4}{\chi_4} = |\lambda|\,, \quad \text{with} \quad \lambda = \frac{(z_1 - z_2)\,(z_3 - z_4)}{(z_1 - z_4)(z_3 - z_2)} = \{z_1, z_2; z_3, z_4\}\,, \tag{3.14}$$

which being independent of the mapping radii, and, by construction a conformal invariant of a four-punctured sphere, necessarily has to equal the cross-ratio of the four points (or a function of the cross-ratio). The cross-ratio, as customary, will be denoted by λ. It is the point where z_1 lands when z_2, z_3 and z_4 are mapped to zero, one and infinity, respectively.

Letting one puncture go to infinity. It is sufficient to consider the behavior of the invariant χ_{12}, given by

$$\chi_{12} \equiv \frac{|z_1 - z_2|^2}{\rho_{D_1(z_1)}\, \rho_{D_2(z_2)}}\,, \tag{3.15}$$

which we have seen is independent of the chosen uniformizer. We must examine what happens as we change the uniformizer in such a way that $z_2 \to \infty$. Given one uniformizer z there is another one $w = 1/z$ that is well defined at $z = \infty$, the only point where z fails to define a local coordinate. This is why there is no naive limit to χ_2 as $z_2 \to \infty$. Using (3.3) we express the mapping radius of the second disk in terms of the mapping radius as viewed using the uniformizer induced by w. We have $\rho_{D_2(z_2)} = \left|\frac{dz}{dw}\right|_{w_2} \rho_{D_2(w_2)} = |z_2|^2 \rho_{D_2(w_2)}$, and substituting into the expression for χ_{12} we find

$$\chi_{12} = \frac{|1 - z_1/z_2|^2}{\rho_{D_1(z_1)}\, \rho_{D_2(w_2)}}\,, \tag{3.16}$$

and we can now take the limit as $z_2 \to \infty$ without difficulty. Writing, for convenience, $\rho_{D_2(w_2=0)} \equiv \rho_{D_2(\infty)}$, we get $\chi_{12} = \rho^{-1}_{D_1(z_1)}\, \rho^{-1}_{D_2(\infty)}$. The apparent dependence of χ_{12} on the choice of point z_1 is fictitious. Any change of uniformizer $z \to az + b$ which changes z_1 leaving the point at infinity fixed, will change the uniformizer at infinity, and the product of mapping radii will remain invariant. The point z_1 can therefore be chosen to be at the origin, and we write our final expression for χ_2

$$\chi_{12} = \frac{1}{\rho_{D_1(0)}\, \rho_{D_2(\infty)}}\,. \tag{3.17}$$

Following exactly the same steps with χ_3 and χ_4 we obtain

$$\chi_{123} \equiv \frac{|z_1 - z_2|}{\rho_{D_1(z_1)}\, \rho_{D_2(z_2)} \rho_{D_3(\infty)}}\,, \tag{3.18}$$

$$\chi_{1234} \equiv \frac{|z_1 - z_2|\, |z_2 - z_3|}{\rho_{D_1(z_1)}\, \rho_{D_2(z_2)} \rho_{D_3(z_3)} \rho_{D_4(\infty)}}\,. \tag{3.19}$$

One could certainly take $z_1 = 0$ and $z_2 = 1$ for χ_{123}, and, $z_2 = 0$ and $z_3 = 1$ for χ_{1234}. It should be remembered that whenever a disk is centered at infinity, the local coordinate used is the inverse of the chosen uniformizer on the rest of the sphere.

3.2. Mapping Radii and Quadratic Differentials

In this subsection we will review how one uses the Strebel quadratic differential on a punctured sphere to define punctured disks. These disks, called coordinate disks, define the local coordinates used to insert the off-shell states. We will show how one can use the quadratic differential to calculate the explicit form of the local coordinates, and the mapping radii of the coordinate disks. We will review the extremal properties of the Strebel quadratic differentials and then discuss the extremal properties of the PSL$(2,\mathbb{C})$ invariants.

We will concentrate on the Strebel quadratic differentials relevant for the restricted polyhedra of closed string field theory. The reader unfamiliar with these objects may consult Refs.[17,21]. The Strebel quadratic differential for a sphere with N punctures in $\mathcal{V}_{0,N}$ induces a metric where the surface can be constructed by gluing N semiinfinite cylinders of circumference 2π across their open boundaries. The gluing pattern is described by a restricted polyhedron, which is a polyhedron having N faces, each of perimeter 2π and, in addition, having all nontrivial closed paths longer than or equal to 2π. Each semiinfinite cylinder defines a punctured disk with a local coordinate w. The boundary $|w| = 1$ corresponds to the edge of the cylinder, to be glued to the polyhedron, and the puncture corresponds to $w = 0$.

The Strebel quadratic differential on the sphere is usually expressed as $\varphi = \phi(z)(dz)^2$, where z is a uniformizer in the sphere. At the punctures it has second order poles; if there is a puncture at $z = z_I$ the quadratic differential near z_I reads

$$\varphi = \Big(-\frac{1}{(z-z_I)^2} + \frac{b_{-1}}{z-z_I} + b_0 + b_1(z-z_I) + \cdots\Big)(dz)^2 \,. \tag{3.20}$$

Moreover, as mentioned above, the quadratic differential defines a disk D_I on the sphere, with the puncture at z_I. A local coordinate w_I on D_I, such that D_I becomes a round disk can be found as follows. We set

$$z = \rho_I\, w_I + c_1 w_I^2 + c_2 w_I^3 + \cdots \,, \tag{3.21}$$

where $\rho_I, c_1, c_2 \cdots$ are constants to be determined. We have written ρ_I for the coefficient of w_I on purpose. If we can make the D_I disk correspond to the disk $|w_I| \leq 1$, then ρ_I is by definition the mapping radius of the disk D_I, since it is the value of $|d(z-z_I)/dw_I|$ at $w_I = 0$ (recall (3.1)). We will actually use the notation

$$z = h_I(w_I), \quad \rightarrow \quad \rho_I = |h_I'(0)| \,. \tag{3.22}$$

Note that as explained in the previous subsection we are using the local uniformizer on the sphere to define the mapping radius.

Back to our problem of defining the w_I coordinate, we demand that the quadratic differential, expressed in w_I coordinates take the form

$$\varphi = -\frac{1}{w_I^2}(dw_I)^2. \tag{3.23}$$

Since the above form is invariant under a change of scale, $w_I \to aw_I$, we cannot determine by this procedure the constant ρ_I. If ρ_I is fixed, the procedure will fix uniquely the higher coefficients $c_1, c_2, \cdots$. While for general off-shell states the knowledge of the coefficients c_I is necessary, for tachyons we only need the mapping radius. This radius can be determined by the following method. Given a quadratic differential one must find an arbitrary point P lying on the boundary of the punctured disk D_I defined by the quadratic differential. Possibly, the simplest way to do this is to identify the zeroes of the quadratic differential and then sketch the critical trajectories to identify the various punctured disks and ring domains. One can then pick P to be a zero lying on the nearest critical trajectory surrounding the puncture. We now require $w_I(P) = 1$, and this will fix both the scale and the phase of the local coordinate. This requirement is satisfied by taking

$$w_I(z) = \exp\Big(i \int_{z(P)}^{z} \sqrt{\phi(\xi)}\, d\xi \Big), \tag{3.24}$$

where we take the positive branch for the square root. If the integral can be done explicitly then the mapping radius is easily calculated by taking a derivative $\rho_I = |\frac{dw_I}{dz}|^{-1}_{z_I}$. If the integral cannot be done explicitly one can calculate the mapping radius by a limiting procedure. One computes $\rho_I = \lim_{\epsilon\to 0} |\frac{dw_I}{dz}|^{-1}_{z_I+\epsilon}$. This leads, using (3.20) to the following result

$$\ln \rho_I = \lim_{\epsilon\to 0} \Bigg(\Im\mathrm{m} \int_{z_I+\epsilon}^{z(P)} \sqrt{\phi(\xi)} d\xi \; + \ln\epsilon \Bigg). \tag{3.25}$$

The integration path is some curve in the disk D_I, and using contour deformation one can verify that the imaginary part of the integrand does not depend on the choice of P as long as P is on the boundary of D_I. When using equation (3.25) one must choose a branch for the square root, and keep the integration path away from the branch cut. The sign is fixed by the condition that the limit exist. Equation (3.25) and the recursive procedure indicated above allow us, in principle, to calculate the function $h_I(w_I)$ if we know explicitly the quadratic differential.

Extremal Properties. Imagine having an N punctured Riemann sphere and label the punctures as $P_1, P_2, \cdots P_N$. Fix completely arbitrary local analytic coordinates at these punctures. Now

consider drawing closed Jordan curves surrounding the punctures and defining punctured disks D_I, in such a way that the disks do not overlap (even though they might touch each other). Given this data we can evaluate the functional

$$\mathcal{F} = M_{D_1} + M_{D_2} + \cdots + M_{D_N} \,, \tag{3.26}$$

which is simply the sum of the reduced moduli of the various disks. This functional, of course depends on the shape of the disks we have chosen, and is well defined since we have picked some specific local coordinates at the punctures. We may try now to vary the shape of the disks in order to maximize $\mathcal{F}$. Suppose there is a choice of disks that maximizes $\mathcal{F}$, then, it will maximize $\mathcal{F}$ whatever choice of local coordinates we make at the punctures. This follows because upon change of local coordinates the reduced modulus of a disk changes by a constant which is independent of the disk itself (see (3.4)). The interesting fact is that the Strebel differential defines the disks that maximize $\mathcal{F}$ [17]. Using the relation between reduced modulus and mapping radius we see that the functional

$$(\rho_{D_1} \cdots \rho_{D_N})^{-1} = \exp(-2\pi\mathcal{F}) \,, \tag{3.27}$$

consisting of the inverse of the product of all the mapping radii, is actually minimized by the choice of disks made by the Strebel quadratic differential. This property will be of use to us shortly.

It is worth pausing here to note that the above definition of the functional $\mathcal{F}$ allows us to compare choices of disks given a *fixed* Riemann sphere. Since we have chosen arbitrarily the local coordinates at the punctures there is no reasonable way to compare the maximal values of $\mathcal{F}$ for two *different* spheres. It is therefore hard to think of Max($\mathcal{F}$) as a function on $\overline{\mathcal{M}}_{0,N}$. This is reminiscent of the fact that while for higher genus surfaces *without* punctures we can think of the area of the minimal area metric as a function on moduli space, it is not clear how to do this for punctured surfaces. The difficulty again is due to the regularization needed to render the area finite, this requires a choice of local coordinates at the punctures, and there is no simple way to compare the choices for different punctured surfaces. We now wish to emphasize that our earlier discussion teaches us how to define functions on $\overline{\mathcal{M}}_{0,N}$. These functions are interesting because they are simple modifications of $\exp[-2\pi\text{Max}(\mathcal{F})]$ that turn out to be functions on $\overline{\mathcal{M}}_{0,N}$.

Indeed, consider the invariant χ_{1234} that was defined as

$$\chi_{1234} \equiv \frac{|z_1 - z_2|\,|z_2 - z_3|\,|z_3 - z_4||z_4 - z_1|}{\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}\rho_{D_3(z_3)}\rho_{D_4(z_4)}} \,. \tag{3.28}$$

Recall that the mapping radii entering in the definition of χ_{1234}, as well as the coordinate differences, are computed using the global uniformizer, and the invariance of χ_{1234} just means

independence of the result on the choice of uniformizer. No choice is required to evaluate χ_{1234}. We obtain a function f on $\overline{\mathcal{M}}_{0,4}$ by giving a number for each four punctured sphere R_4 as follows. We equip the sphere R_4 with the Strebel quadratic differential $\varphi_S(R_4)$ and we evaluate the invariant χ_{1234} using the disks $D^I[\varphi_S(R_4)]$ determined by the differential. In writing

$$f(R_4) \equiv \chi_{1234}\left(D^I\left[\varphi_S(R_4)\right]\right). \tag{3.29}$$

We claim that $f(R_4)$ actually is the lowest value that the invariant χ_{1234} can take for any choice of nonoverlapping disks in R_4

$$\chi_{1234}\left(D^I\left[\varphi_S(R_4)\right]\right) \leq \chi_{1234}\left(D^I\left[R_4\right]\right). \tag{3.30}$$

To see this, fix a uniformizer such that three of the punctures lie at three points (say, $z = -1, 0, 1$) and the fourth puncture will lie at some fixed point, which depends on the choice of four punctured sphere. This fixes completely the numerator of χ and fixes the local coordinates at the punctures, necessary to compute the mapping radii. Therefore

$$\chi_{1234} \propto \left(\rho_{D_1(z_1)}\,\rho_{D_2(z_2)}\rho_{D_3(z_3)}\rho_{D_4(z_4)}\right)^{-1} = \exp(-2\pi\mathcal{F})\,, \tag{3.31}$$

where we recognize that, up to a fixed constant, the invariant is simply related to the value of $\mathcal{F}$ evaluated with the chosen coordinates at the punctures. As we now vary the disks around the punctures, $\mathcal{F}$ will be maximized by the quadratic differential. This verifies that χ is minimized by the disks chosen by the quadratic differential (Eqn.(3.30).)

We expect the function $f(R_4)$ to have a minimum for the most symmetric surface in $\mathcal{M}_{0,4}$, namely, for the regular tetrahedron $[\lambda = (1+i\sqrt{3})/2]$. We have not proven this, but the intuition is that for the most symmetric surface we can get the disks of largest mapping radii. There is, of course the issue of the numerator of χ with the coordinate differences, which also varies as we move in moduli space. Still, one can convince oneself that the function $f(R_4)$ grows without bound as R_4 approaches degeneration.

Estimating Mapping Radii. As we have mentioned earlier, in defining the mapping radius of a punctured disk on the sphere we use a local coordinate at the puncture which is obtained from a chosen uniformizer on the sphere. While this mapping radius depends on the uniformizer, we are typically interested in functions, such as the χ functions, which are constructed out of mapping radii and coordinate differences, and are independent of the chosen uniformizer.

Consider now the sphere as the complex plane z together with the point at infinity. The two following facts are useful tools to estimate the mapping radius of a punctured disk centered at z_0.

(i) If the disk D is actually a round disk $|z-z_0| \leq R$, then the mapping radius ρ_D is precisely given by the radius of the disk: $\rho_D = R$. This is clear since $w = (z-z_0)/R$ is the exact conformal map of D to a unit disk.

(ii) If the disk D is not round but it is contained between two round disks centered at z_0 with radii R_1 and R_2, with $R_1 < R_2$, then $R_1 < \rho_{_D} < R_2$. This property follows from the superadditivity of the reduced modulus (see Ref.[22] Eqn. (2.2.25)).

Given an N punctured sphere, the Strebel quadratic differential will maximize the product of the N mapping radii. We can obtain easily a bound $\rho_1\rho_2\cdots\rho_{_N} \geq R_1 R_2 \cdots R_{_N}$, where the $R_{_I}$ are the radii of non-overlapping round disks centered at the punctures with the sphere represented as the complex plane together with the point at infinity.

4. Off-Shell Amplitudes for Tachyons

In this section we compute off-shell amplitudes for tachyons at arbitrary momentum. We first discuss the case of three tachyons and then the case of $N \geq 4$ tachyons which requires integration over moduli space. We examine the results for the case of zero-momentum tachyons obtaining in this way the coefficients $\mathrm{v}_{_N}$ of the tachyon potential. We explain why the choice of polyhedra for the string vertices, minimizes recursively the coefficients of the nonpolynomial tachyon potential.

4.1. Three Point Couplings

We will now examine the cubic term in the string field potential. Assume we are now given a three punctured sphere, and we want to calculate the general off-shell amplitude for three tachyons. We then must compute the correlator

$$A_{p_1p_2p_3} = \Big\langle c\bar{c}e^{ip_1X}(w_1=0)\; c\bar{c}e^{ip_2X}(w_2=0)\; c\bar{c}e^{ip_3X}(w_3=0)\Big\rangle\,. \tag{4.1}$$

We must transform these operators from the local coordinates $w_{_I}$ to some uniformizer z. Let $w_{_I}=0$ correspond to $z = z_{_I}$. We then have from the transformation law of a primary field

$$c\bar{c}e^{ip_IX}(w_{_I}=0) = c\bar{c}e^{ip_IX}(z=z_{_I})\Big|\frac{dz}{dw_{_I}}\Big|^{p_I^2-2}_{w_I=0} = c\bar{c}e^{ip_IX}(z_{_I})\;\rho_1^{p_I^2-2}\,, \tag{4.2}$$

where in the last step we have recognized the appearance of the mapping radius for the disk $D_{_I}$. The correlator then becomes

$$\begin{aligned} A_{p_1p_2p_3} &= \Big\langle c\bar{c}e^{ipX}(z_1)\; c\bar{c}e^{ipX}(z_2)\; c\bar{c}e^{ipX}(z_3)\Big\rangle \frac{1}{\rho_1^{2-p_1^2}\rho_2^{2-p_2^2}\rho_3^{2-p_3^2}}\,,\\ &= \frac{|z_1-z_2|^{2+2p_1p_2}|z_2-z_3|^{2+2p_2p_3}|z_1-z_3|^{2+2p_1p_3}}{\rho_1^{2-p_1^2}\rho_2^{2-p_2^2}\rho_3^{2-p_3^2}}\cdot[-2(2\pi)^D\delta^D(\textstyle\sum p_{_I})\,]\,, \end{aligned} \tag{4.3}$$

where we made use of (2.7), which introduces an extra factor of -2 (shown in brackets) due to our convention $c_0^{\pm} = (c_0 \pm \bar{c}_0)/2$. In order to construct a manifestly PSL(2,$\mathbb{C}$) invariant

expression we use momentum conservation in the denominator to write

$$\begin{aligned} A_{p_1p_2p_3} &= \frac{|z_1-z_2|^{2+2p_1p_2}}{(\rho_1\rho_2)^{1+p_1p_2}}\,\frac{|z_2-z_3|^{2+2p_2p_3}}{(\rho_2\rho_3)^{1+p_2p_3}}\,\frac{|z_1-z_3|^{2+2p_1p_3}}{(\rho_1\rho_3)^{1+p_1p_3}}\cdot\big[-2(2\pi)^D\delta^D(\textstyle\sum p_{\scriptscriptstyle I})\,\big]\,, \\ &= \big[-2(2\pi)^D\delta^D(\textstyle\sum p_{\scriptscriptstyle I})\,\big]\cdot\displaystyle\prod_{I<J}^{3}\,[\chi_{\scriptscriptstyle IJ}]^{1+p_{\scriptscriptstyle I}p_{\scriptscriptstyle J}}\,, \end{aligned} \tag{4.4}$$

which is the manifestly PSL(2,$\mathbb{C}$) invariant description of the off-shell amplitude.

We can now use the above result to extract the cubic coefficient of the tachyon potential. By definition, the operator formalism bra $\langle V^{(3)}_{123}|$ representing the three punctured sphere must satisfy

$$\langle V^{(3)}_{123}|\,(c_1\bar c_1|\mathbf{1},p_1\rangle)^{(1)}\,(c_1\bar c_1|\mathbf{1},p_2\rangle)^{(2)}\,(c_1\bar c_1|\mathbf{1},p_3\rangle)^{(3)}\;=A_{p_1p_2p_3}\,. \tag{4.5}$$

Moreover, the multilinear function representing the cubic interaction is given by

$$\begin{aligned} \{T\}_{\mathcal{V}_{0,3}} &\equiv \langle V^{(3)}_{123}|T\rangle^{(1)}|T\rangle^{(2)}|T\rangle^{(3)}\,, \\ &= \int\frac{\mathrm{d}p_1}{(2\pi)^D}\frac{\mathrm{d}p_2}{(2\pi)^D}\frac{\mathrm{d}p_3}{(2\pi)^D}\,A_{p_1p_2p_3}\,\tau(p_1)\tau(p_2)\tau(p_3)\,, \\ &= \int\prod_{I=1}^{3}\frac{\mathrm{d}p_{\scriptscriptstyle I}}{(2\pi)^D}\,(2\pi)^D\delta^D(\textstyle\sum p_{\scriptscriptstyle I})\;\cdot(-2)\displaystyle\prod_{I<J}^{3}\,[\chi_{\scriptscriptstyle IJ}]^{1+p_{\scriptscriptstyle I}p_{\scriptscriptstyle J}}\;\cdot\tau(p_1)\tau(p_2)\tau(p_3)\,, \end{aligned} \tag{4.6}$$

where use was made of the definition of the string tachyon field in (2.2), of (4.5), and of (4.4). Comparison with (2.10), and use of (2.13) now gives

$$\mathrm{v}_{\scriptscriptstyle 3} = -2\cdot\prod_{I<J}^{3}\,[\chi_{\scriptscriptstyle IJ}] = -2\cdot\frac{|z_1-z_2|^2|z_2-z_3|^2|z_1-z_3|^2}{\rho_1^2\rho_2^2\rho_3^2} = -2\cdot\chi^2_{123}\,, \tag{4.7}$$

in terms of the PSL(2,$\mathbb{C}$) invariant χ_{123}. It follows from the extremal properties discussed earlier that the minimum value possible for $\mathrm{v}_{\scriptscriptstyle 3}$ is achieved for the Strebel quadratic differential defining the Witten vertex. We will calculate the minimum possible value for $\mathrm{v}_{\scriptscriptstyle 3}$ in sect.5.1.

4.2. The Off-Shell Koba-Nielsen Formula

We now derive a formula for the off-shell scattering amplitude for N closed string tachyons at arbitrary momentum. The final result will be a manifestly PSL(2,$\mathbb{C}$) invariant expression. The computation is simplified because the tachyon vertex operator is primary even off-shell, and because its ghost structure is essentially trivial.

We work in the z-plane and fix the position of the three last insertions at z_{N-2}, z_{N-1} and z_N. The positions of the first $N-3$ punctures will be denoted as $z_1, z_2, \cdots z_{N-3}$. We must integrate over the positions of these $N-3$ punctures. Each will therefore give a factor

$$\mathrm{d}x_I \wedge \mathrm{d}y_I \; b(\frac{\partial}{\partial x})\, b(\frac{\partial}{\partial y}) = \mathrm{d}x_I \wedge \mathrm{d}y_I \;\; 2i\, \bar{b}_{-1} b_{-1} = -\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_I \, \bar{b}_{-1} b_{-1} \,, \tag{4.8}$$

where $z_I = x_I + iy_I$. There is a subtlety here, each of the antighost oscillators refers to the z plane, while the ghost oscillators in each tachyon insertion $c_1^{w_I} \bar{c}_1^{w_I} |0,p\rangle^{w_I}$ refer to the local coordinate w_I, where $z = h_I(w_I)$. Transforming the antighost oscillators we obtain $b_{-1} = [h_I'(0)]^{-1} b_{-1}^{w_I} + \cdots$, where the dots indicate antighost oscillators $b_{n\geq 0}^{w_I}$ that annihilate the tachyon state. For the antiholomorphic oscillator we have $\bar{b}_{-1} = \left[\overline{h_I'(0)}\right]^{-1} \bar{b}_{-1}^{w_I} + \cdots$. Therefore each of the integrals will be represented by

$$2i \,\mathrm{d}x_I \wedge \mathrm{d}y_I \; \frac{1}{\rho_I^2} \,|0,p\rangle^{w_I} = 2i \,\mathrm{d}x_I \wedge \mathrm{d}y_I \; \frac{1}{\rho_I^{2-p_I^2}} \,|0,p\rangle \,, \tag{4.9}$$

where $\rho_I = |h_I'(0)|$ is the mapping radius of the I-th disk. The Koba-Nielsen amplitude will therefore be given by

$$\begin{aligned} A_{p_1\cdots p_N} = \left(\frac{i}{2\pi}\right)^{N-3} (2i)^{N-3} \int \prod_{I=1}^{N-3} \frac{\mathrm{d}x_I \mathrm{d}y_I}{\rho_I^{2-p_I^2}} \frac{1}{\rho_{N-2}^{2-p_{N-2}^2}\, \rho_{N-1}^{2-p_{N-1}^2}\, \rho_{N-2}^{2-p_{N-2}^2}} \\ \cdot \left\langle e^{ip_1 X(z_1)} \cdots e^{ip_{N-3} X(z_{N-3})} \; c\bar{c} e^{ip_{N-2} X(z_{N-2})} \; c\bar{c} e^{ip_{N-1} X(z_{N-1})} \; c\bar{c} e^{ip_N X(z_N)} \right\rangle , \end{aligned} \tag{4.10}$$

where the correlator is a free-field correlator in the complex plane. We will not include in the amplitude the coupling constant factor κ^{N-3}. The extra factor $(i/2\pi)^{N-3}$ included in the formula above is well-known to be necessary for consistent factorization, and has been derived in closed string field theory.* We then have

$$\begin{aligned} A_{p_1\cdots p_N} = (-)^N \cdot \frac{2}{\pi^{N-3}} \int \prod_{I=1}^{N-3} \frac{\mathrm{d}x_I \mathrm{d}y_I}{\rho_I^{2-p_I^2}} \; \frac{|(z_{N-2} - z_{N-1})\,(z_{N-2} - z_N)\,(z_{N-1} - z_N)|^2}{\rho_{N-2}^{2-p_{N-2}^2}\, \rho_{N-1}^{2-p_{N-1}^2}\, \rho_N^{2-p_N^2}} \\ \cdot \prod_{I<J}^{N} |z_I - z_J|^{2p_I p_J} \cdot \left[(2\pi)^D \delta\left(\sum p_I\right)\right] . \end{aligned} \tag{4.11}$$

$\star$ The value used here appears in Ref.[2], where a sign mistake of Ref. [18] was corrected.

Using momentum conservation, and the definition of the invariants χ_{IJ} and χ_{IJK}, we can write the above as

$$A_{p_1\cdots p_N} = (-)^N \frac{2}{\pi^{N-3}} \int \prod_{I=1}^{N-3} \frac{\mathrm{d}x_I \mathrm{d}y_I}{\rho_I^2}\ \chi^2_{N-2,N-1,N} \cdot \prod_{I<J}^{N} \chi_{IJ}^{p_I p_J} \cdot \Big[(2\pi)^D \delta\Big(\sum p_I\Big)\Big], \tag{4.12}$$

It follows immediately from the transformation law for the mapping radius that the measure $dz_I \wedge d\bar{z}_I/\rho_I^2$ is PSL(2,$\mathbb{C}$) invariant. Therefore the above result is a manifestly PSL(2,$\mathbb{C}$) invariant off-shell generalization of the Koba-Nielsen formula. For the case of four tachyons it reduces to an off-shell version of the Virasoro-Shapiro amplitude

$$A_{p_1\cdots p_4} = \frac{2}{\pi} \int \frac{\mathrm{d}x_1 \mathrm{d}y_1}{\rho_1^2}\ \chi^2_{234} \cdot \prod_{I<J}^{4} \chi_{IJ}^{p_I p_J} \cdot \Big[(2\pi)^D \delta\Big(\sum p_I\Big)\Big]. \tag{4.13}$$

If we choose to place the second, third, and fourth punctures at zero, one and infinity respectively, we end with

$$A_{p_1\cdots p_4} = \frac{2}{\pi} \int \mathrm{d}x\, \mathrm{d}y\, \frac{|z|^{2p_1p_2}|z-1|^{2p_1p_3}}{\rho_1^{2-p_1^2}\rho_2^{2-p_2^2}\rho_3^{2-p_3^2}\rho_4^{2-p_4^2}} \cdot \Big[(2\pi)^D \delta\Big(\sum p_I\Big)\Big]. \tag{4.14}$$

Another expression can be found where the variables of integration are cross-ratios. We define the cross ratio

$$\lambda_I \equiv \{z_I, z_{N-2}; z_{N-1}, z_N\} = \frac{(z_I - z_{N-2})\,(z_{N-1} - z_N)}{(z_I - z_N)\,(z_{N-1} - z_{N-2})}, \tag{4.15}$$

and it follows that

$$\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_I = \frac{\mathrm{d}\lambda_I \wedge \mathrm{d}\bar{\lambda}_I}{|\lambda_I|^2} \frac{|z_I - z_N|^2 |z_I - z_{N-2}|^2}{|z_N - z_{N-2}|^2}, \tag{4.16}$$

leading to

$$\begin{aligned} A_{p_1\cdots p_N} = 2(-)^N \left(\frac{i}{2\pi}\right)^{N-3} \int \prod_{I=1}^{N-3} \Big[\frac{\mathrm{d}\lambda_I \wedge \mathrm{d}\bar{\lambda}_I}{|\lambda_I|^2} \Big(\frac{\chi_{I,N-2,N-1,N}}{\chi_{N-2,N-1,N}}\Big)^2\Big] \cdot \chi^2_{N-2,N-1,N} \\ \cdot \prod_{I<J}^{N} \chi_{IJ}^{p_I p_J} \cdot \Big[(2\pi)^D \delta\Big(\sum p_I\Big)\Big]. \end{aligned} \tag{4.17}$$

For the case of four tachyons the above result reduces to

$$A_{p_1\cdots p_4} = \frac{i}{\pi} \int \frac{\mathrm{d}\lambda \wedge \mathrm{d}\bar{\lambda}}{|\lambda|^2}\, \chi^2_{1234} \cdot \prod_{I<J}^{4} \chi_{IJ}^{p_I p_J} \cdot \Big[(2\pi)^D \delta\Big(\sum p_I\Big)\Big]. \tag{4.18}$$

In the above expressions the λ integrals extend over the whole sphere.

Having obtained manifestly PSL(2,$\mathbb{C}$) invariant expressions valid for arbitrary momenta, we now go back to our particular case of interest, which is the case when all the momenta are zero. It is simplest to go back to (4.10) to obtain

$$A_{1\cdots N} = (-)^N \frac{2}{\pi^{N-3}} \int \prod_{I=1}^{N-3} \frac{\mathrm{d}x_{I}\mathrm{d}y_{I}}{\rho_{I}^2} \; \frac{|z_{N-2}-z_{N-1}|^2 |z_{N-2}-z_{N}|^2 |z_{N-1}-z_{N}|^2}{\rho_{N-2}^2 \rho_{N-1}^2 \rho_{N}^2} [(2\pi)^D \delta(\mathbf{0})] \,, \tag{4.19}$$

$$A_{1\cdots 4} = \frac{2}{\pi} \int \frac{\mathrm{d}x_1\, \mathrm{d}y_1}{\rho_1^2} \; \frac{|z_2-z_3|^2 |z_2-z_4|^2 |z_3-z_4|^2}{\rho_2^2 \rho_3^2 \rho_4^2} [(2\pi)^D \delta(\mathbf{0})] \,, \tag{4.20}$$

These are the expressions we shall be trying to estimate. If we set the three special points appearing in the above expressions to zero, one and infinity, we find (see sect. 3.3)

$$A_{1\cdots N} = (-)^N \frac{2}{\pi^{N-3}} \int \prod_{I=1}^{N-3} \frac{\mathrm{d}x_{I} dy_{I}}{\rho_{I}^2} \; \frac{1}{\rho_{N-2}^2(0) \rho_{N-1}^2(1) \rho_{N}^2(\infty)} \, [(2\pi)^D \delta(\mathbf{0})] \,, \tag{4.21}$$

$$A_{1\cdots 4} = \frac{2}{\pi} \int \frac{\mathrm{d}x_1\, \mathrm{d}y_1}{\rho_1^2} \; \frac{1}{\rho_2^2(0) \rho_3^2(1) \rho_4^2(\infty)} \, [(2\pi)^D \delta(\mathbf{0})] \,. \tag{4.22}$$

Let us now use the above results to extract the quartic and higher order coefficient of the tachyon potential. By definition, the operator formalism bra representing the collection of N-punctured spheres must satisfy

$$\int_{\mathcal{V}_{0,N}} \langle \Omega_N^{(0)0,N} | \, (c_1\bar{c}_1 |\mathbf{1}, p_1\rangle)^{(1)} \cdots (c_1\bar{c}_1 |\mathbf{1}, p_N\rangle)^{(N)} \; = A_{p_1\cdots p_N}(\mathcal{V}_{0,N}) \,, \tag{4.23}$$

where the $\mathcal{V}_{0,N}$ argument of $A_{p_1\cdots p_N}(\mathcal{V}_{0,N})$ indicates that the off-shell amplitude has only been partially integrated over the subspace $\mathcal{V}_{0,N}$. The corresponding multilinear function is given by

$$\{T\}_{\mathcal{V}_{0,N}} \equiv \int_{\mathcal{V}_{0,N}} \langle \Omega_N^{(0)0,N} | T\rangle^{(1)} \cdots |T\rangle^{(N)} \; = \int \prod_{I=1}^{N} \frac{\mathrm{d}p_{I}}{(2\pi)^D} \; A_{p_1\cdots p_N}(\mathcal{V}_{0,N}) \, \tau(p_1) \cdots \tau(p_N) \,, \tag{4.24}$$

where we made use of the definition of the string tachyon field in (2.2), and of (4.23). Reading

the value of the amplitude at zero momentum, and by virtue of (2.10) and (2.13) we get

$$\begin{aligned}
\mathrm{v}_{N} &= (-)^{N} \frac{2}{\pi^{N-3}} \int_{\mathcal{V}_{0,N}} \prod_{I=1}^{N-3} \frac{\mathrm{d}x_{I}\mathrm{d}y_{I}}{\rho_{I}^{2}} \; \frac{|z_{N-2}-z_{N-1}|^{2}|z_{N-2}-z_{N}|^{2}|z_{N-1}-z_{N}|^{2}}{\rho_{N-2}^{2}\rho_{N-1}^{2}\rho_{N}^{2}}\,, \\
&= (-)^{N} \frac{2}{\pi^{N-3}} \int_{\mathcal{V}_{0,N}} \prod_{I=1}^{N-3} \frac{\mathrm{d}x_{I}dy_{I}}{\rho_{I}^{2}} \; \frac{1}{\rho_{N-2}^{2}(0)\rho_{N-1}^{2}(1)\rho_{N}^{2}(\infty)}\,.
\end{aligned} \tag{4.25}$$

Note here the pattern of signs. All v_{N} for N even come with positive sign, and all v_{N} for N odd come with a negative sign. Including the overall minus sign in passing from the action to the potential (Eqn.(2.12)), and the sign redefinition $\tau \to -\tau$, all coefficients of the tachyon potential become negative.

It is worthwhile to pause and reflect about the above pattern of signs. In particular, since v_{4} turned out to be positive, the quartic term in the tachyon potential is negative, as the quadratic term is. While the calculations leading to the sign factors are quite subtle, we believe that the result should have been expected. In closed string field theory, the elementary four point interaction changes if we include stubs in the three string vertex. Both the original interaction and the one for the case of stubs must have the same sign, because they only differ by the region of integration over moduli space, and the integrand, as we have seen, has a definite sign. On the other hand, the interaction arising from the stubbed theory would equal the original interaction plus a collection of Feynman graphs with two three-string vertices and with one propagator whose proper time is only partially integrated. Such terms, for completely integrated propagators and massive intermediate fields would give a contribution leading to a potential unbounded below. For partially integrated propagators they also contribute such kind of terms, both if the field is truly massive, or if it is tachyonic. This indicates that one should have expected an unbounded below elementary interaction.

It is now simple to explain why the choice of restricted polyhedra (polyhedra with all nontrivial closed paths longer than or equal to 2π [23]) for closed string vertices minimizes recursively the expansion coefficients of the tachyon potential. We have seen that v_{3} is minimized by the Witten vertex. At the four point level we then have a missing region $\mathcal{V}_{0,4}$. In the parametrization given by the final form in (4.25) the region of integration corresponding to $\mathcal{V}_{0,4}$ is fixed. At each point in this region, the integrand, up to a constant, is given by $1/\prod_{i}\rho_{i}^{2} = \exp(-4\pi\mathcal{F})$, and as explained around Eqn.(3.27), this quantity is minimized by the choice of coordinate disks determined by the Strebel differential. Since the integrand is positive definite throughout the region of integration, and, at every point is minimized by the use of the Strebel differential, it follows that the integral is minimized by the choice of the Strebel differential for the string vertex. That is precisely the choice that defines the restricted polyhedron corresponding to the standard four closed string vertex. It is clear that the above considerations hold for any $\mathcal{V}_{0,N}$. The minimum value for v_{N} is obtained by using polyhedra

throughout the region of integration. Therefore, starting with the three string vertex, we are led recursively by the minimization procedure to the restricted polyhedra of closed string field theory.

5. Estimates for the Tachyon Potential

The present section is devoted to estimates of the tachyon potential. While more analytic work on the evaluation of the tachyon potential may be desirable, here we will get some intuitive feeling for the growth of the coefficients $\mathrm{v}_{_N}$ for large N. The aim is to find if the tachyon potential has a radius of convergence. We will not be able to decide on this point, but we will obtain a series of results that go in this direction.

For every number of punctures N, there is a configuration of these punctures on the sphere for which we can evaluate exactly the measure of integration. This is the configuration where the punctures are "equally separated" in a planar arrangement. These configurations appear as a finite number of points in the boundary of $\mathcal{V}_{0,N}$, and in some sense are the most problematic. The large N behavior of the measure at those points is such that if the whole integrand were to be dominated by these points the tachyon potential would seem to have no radius of convergence. The shape of $\mathcal{V}_{0,N}$ around those points, however, is such that the contributions might be suppressed.

In each $\mathcal{V}_{0,N}$ there are configurations where the punctures are distributed most symmetrically. It is intuitively clear that at these configurations the measure is in some sense lowest. It is possible to estimate this measure for large N, and conclude that, if dominated by this contribution, the tachyon potential should have some radius of convergence.

The behavior of the measure for the tachyon potential is such that the measure grows as we approach degeneration, and if $\mathcal{V}_{0,4}$, for example, was to extend over all of $\overline{\mathcal{M}}_{0,4}$ the naive integral would be infinite. This infinity is not physical, because we do not expect infinite amplitude for the scattering of four zero-momentum tachyons. We explain how analytic continuation of the contribution from the Feynman graphs removes this apparent contradiction.

We begin by presenting several exact results pertaining two, three, and four-punctured spheres.

5.1. Evaluation of Invariants

Consider first the invariant χ_{12} of a sphere with two punctured disks (Eqn.(3.7)). The disks may touch but they are assumed not to overlap. Since the mapping radii can be as small as desired, the invariant χ_{12} is not bounded above. It is actually bounded below, by the value attained when we have a Strebel quadratic differential. We can take the sphere punctured at zero and infinity, and the quadratic differential to be $\varphi = -(dz)^2/z^2$. While this differential does not determine a critical trajectory, we can take it to be any closed horizontal trajectory,

say $|z| = 1$. This divides the z-sphere into two disks, one punctured at $z = 0$ with unit disk $|z| \leq 1$ and the other punctured at $z = \infty$, or $w = 0$, with $w = 1/z$, and with unit disk $|w| \leq 1$. It follows that their mapping radii are both equal to one. We therefore have, using (3.17)

$$\widehat{\chi}_{12} \equiv \mathrm{Min}\,(\chi_{12}) = \mathrm{Min}\Big(\frac{1}{\rho_{D_1(0)}\,\rho_{D_2(\infty)}}\Big) = 1\,. \tag{5.1}$$

We now consider three punctured spheres and try to evaluate the minimum value of the invariant χ_{123} defined in Eqn.(3.10). This minimum is achieved for the three punctured sphere corresponding to the Witten vertex. In this vertex we can represent the sphere by the z plane, with the punctures at $e^{\pm i\pi/3}, -1$. The disk surrounding the puncture at $z = e^{i\pi/3}$ is the wedge domain $0 \leq \mathrm{Arg}(z) \leq 2\pi/3$. This domain is mapped to a unit disk $|w| \leq 1$ by the transformations

$$t = z^{3/2}\,, \quad w = \frac{t-i}{t+1}\,. \tag{5.2}$$

One readily finds that the mapping radius ρ of the disk is given by $\rho = |\frac{dz}{dw}|_{w=0} = 4/3$. Furthermore, the distance $|z_I - z_J|$ between any of the punctures is equal to $\sqrt{3}$. Therefore back in (3.10) we obtain

$$\widehat{\chi}_{123} \equiv \mathrm{Min}\,(\chi_{123}) = \Big(\frac{3\sqrt{3}}{4}\Big)^3\,.$$

This result implies that the minimum possible value for $|\mathrm{v}_3|$ is realized with $\mathrm{v}_3 = -3^9/2^{11}$ (see Eqn.(4.7)).

Another computation that is of interest is that of the most symmetric four punctured sphere, a sphere where the punctures are at $z = 0, 1, \infty$, and at $z = \rho = e^{i\pi/3}$. The Strebel quadratic differential for this sphere can be found to be

$$\varphi = \frac{(2z+1)(2z-\rho)(2z-\bar{\rho})}{(z-1)^2(z-\rho^2)^2(z-\bar{\rho}^2)^2} \cdot \frac{(dz)^2}{z^2}\,. \tag{5.3}$$

Here the poles are located at the points $z = 0,\ 1,\ \rho^2$, and $\bar{\rho}^2$. The zeroes are located at $z = -\frac{1}{2},\ \frac{1}{2}\rho,\ \frac{1}{2}\bar{\rho}$, and ∞. One can use this expression for a calculation of the mapping radii.

5.2. The Measure at the Planar Configuration

In each $\mathcal{V}_{0,N}$, for $N \geq 4$ there is a set of symmetric planar configurations for the punctures. They correspond to the surfaces obtained by Feynman diagrams constructed using only the three string vertex, and with all the propagators collapsed with zero twist angle. We will consider the case of N punctures and give an exact evaluation for the measure. This will be done in the frame where three punctures are mapped to the standard points $z = 0, 1, \infty$, and the rest of the punctures will be mapped to the points $z_1, z_2, \cdots z_{N-3}$ lying on the real line in between $z = 0$ and $z = 1$. Consequently, we will take $z_{N-2} = 1, z_{N-1} = \infty$ and $z_N = 0$. The measure we will calculate will be defined as

$$d\mu_N \equiv \prod_{I=1}^{N-3} \frac{\mathrm{d}x_I \mathrm{d}y_I}{\rho_I^2} \; \frac{|z_{N-2} - z_{N-1}|^2 |z_{N-2} - z_N|^2 |z_{N-1} - z_N|^2}{\rho_{N-2}^2 \rho_{N-1}^2 \rho_N^2}\,, \tag{5.4}$$

which, up to constants, is the measure that appears in (4.19). The result will be of the form $d\mu_N = f_N \prod_{k=1}^{N-3} \mathrm{d}x_k \mathrm{d}y_k$, where f_N is a number depending on the number of punctures.

We begin the computation by using a ξ plane where we place all the N punctures equally spaced on the unit circle $|\xi| = 1$. We thus let ξ_k be the position of the k-th puncture, with

$$\xi_k = \exp\bigl((2k-1)i\pi/N\bigr)\,, \quad k = 1, 2, \cdots N\,. \tag{5.5}$$

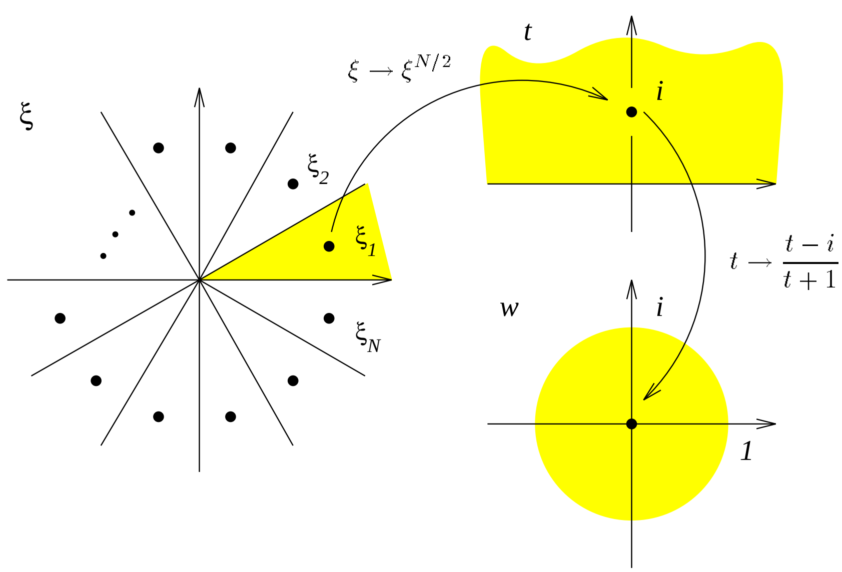

Figure 1. Planar configuration of punctures on the sphere. Shown are the maps from the ring domain associated with a specific puncture to the unit disk.

In this presentation the ring domain surrounding a puncture, say the first one, is the wedge domain $0 \leq \mathrm{Arg}(\xi) \leq 2\pi/N$ (see Figure 1) The mapping radius can be computed exactly by mapping the wedge to the unit disk $|w| \leq 1$, via $t = \xi^{N/2}$ and $w = \frac{t-i}{t+1}$. The result is $\rho = 4/N$, and picking the three special punctures to be ξ_{N-2}, ξ_{N-1} and ξ_N, we find

$$\mathrm{d}\mu_N = 64 \cdot \sin^4\Big(\frac{\pi}{N}\Big) \cdot \sin^2\Big(\frac{4\pi}{N}\Big) \cdot \Big(\frac{N}{4}\Big)^{2N} \cdot \prod_{k=1}^{N-3} \mathrm{d}^2\xi_k \,, \tag{5.6}$$

where $\mathrm{d}^2\xi_k = \mathrm{dRe}\,\xi_k \,\mathrm{dIm}\,\xi_k$. This is the measure, but in the ξ plane. In order to transform it to the z-plane we need the PSL(2,$\mathbb{C}$) transformation that will satisfy $z(\xi_N) = 0, z(\xi_{N-1}) = 1$, and $z(\xi_{N-2}) = \infty$. The desired transformation is

$$z = \frac{\xi - e^{-i\pi/N}}{\xi - e^{-3i\pi/N}} \cdot \beta\,, \quad \text{with} \quad |\beta| = \frac{\sin(\pi/N)}{\sin(2\pi/N)}\,, \tag{5.7}$$

and a small calculation gives

$$\mathrm{d}^2\xi_k = \Big|\frac{dz}{d\xi}\Big|_{\xi_k}^{-2} \mathrm{d}x_k\mathrm{d}y_k = 4 \cdot \frac{\sin^2(2\pi/N)}{\sin^4(\pi/N)} \cdot \sin^4\Big(\frac{(k+1)\pi}{N}\Big) \cdot \mathrm{d}x_k\mathrm{d}y_k\,. \tag{5.8}$$

This expression, used in (5.6) gives us the desired expression for the measure

$$\begin{aligned}\mathrm{d}\mu_N = \; & 4^{2N-3} \cdot \sin^4\Big(\frac{\pi}{N}\Big) \cdot \sin^2\Big(\frac{4\pi}{N}\Big) \cdot \Big(\frac{N}{4}\Big)^{2N} \cdot \Big(\frac{\cos(\pi/N)}{\sin(\pi/N)}\Big)^{2N-6} \\ & \cdot \Big[\prod_{k=1}^{N-3} \sin^4\Big(\frac{(k+1)\pi}{N}\Big)\Big] \cdot \prod_{k=1}^{N-3} \mathrm{d}x_k\,\mathrm{d}y_k\,.\end{aligned} \tag{5.9}$$

This is an exact result, valid for all $N \geq 4$. For the case of $N = 4$ it gives $\mathrm{d}\mu_4 = 256\mathrm{d}x\mathrm{d}y$.⋆ Let us now consider the leading behavior of this measure as $N \to \infty$. The only term that requires some calculation is the product $a_N \equiv \prod_{k=1}^{N-3} \sin^4((k+1)\pi/N)$. One readily finds that as $N \to \infty$

$$\ln a_N \sim 4 \cdot \frac{N}{\pi} \int_0^\pi d\theta \ln(\sin\theta) = -4N\ln(2) \quad \Rightarrow \quad a_N \sim 4^{-2N}, \tag{5.10}$$

⋆ For $N = 4$ the measure can also be calculated exactly for the configuration with cross ratio equal to $(1 + i\sqrt{3})/2$. One finds [19] $\mathrm{d}\mu_4 = \frac{2^{11}}{3^4\sqrt[3]{2}}\mathrm{d}x\,\mathrm{d}y \approx 20.07\,\mathrm{d}x\,\mathrm{d}y$. This correponds to the measure at the "center" of $\mathcal{V}_{0,4}$ and is indeed much smaller than the measure $256\,\mathrm{d}x\,\mathrm{d}y$ at the corners of $\mathcal{V}_{0,4}$.

and using this result, we find the large N behaviour of the planar measure

$$
d\mu_{N} \sim 4^5 \cdot \pi^6 \cdot \left[\frac{N^2}{4\pi}\right]^{2N-6} \cdot \prod_{k=1}^{N-3} dx_k dy_k \,. \tag{5.11}
$$

This was the result we were after. We see that this measure grows like N^{4N}. This growth is so fast that presents an obstruction to a simple proof of convergence for the series defining the tachyon potential. Indeed, a very naive estimation would not yield convergence. Let us see this next.

Let us assume that this planar uniform configuration is indeed the point in $\mathcal{V}_{0,N}$ for which the measure is the largest. This statement requires explanation, since the numerical coefficient appearing in front of a measure can be changed by PSL$(2,\mathbb{C})$ transformations. Thus given any other configuration in $\mathcal{V}_{0,N}$ with a puncture at $0, 1$ and ∞ we do a transformation $z \to az$ with $a = 1/z_{\max}$, where $z_{\max}$ is the position of the puncture farther away from the origin. In this way we obtain a configuration with all the punctures in the unit disk, the same two punctures at zero and infinity, and some puncture at one. At this point the measures can be compared and we expect the planar one to be larger. It is now clear from the construction that the full $\mathcal{M}_{0,N}$ is overcounted if we fix two punctures, one at zero, the other at infinity, and among the rest we pick one at a time to be put at one, while the others are integrated all over the inside of the unit disk. If we estimate this integral using our value for the measure in the worst configuration we get that each coefficient $\mathrm{v}_{N} < [(N-2)\pi^{N-3}]N^{4N-12}$, where the prefactor in brackets arises from the above described integrals (this prefactor does not really affect the issue of convergence). The growth N^{4N} rules out the possibility of convergence. This bound is quite naive, but raises the possibility that there may be no radius of convergence for the tachyon potential.

5.3. The Most Uniform Distribution of Punctures

The corners of $\mathcal{V}_{0,N}$ turned out to be problematic. Since we expect the measure for the coefficients of the tachyon potential to be lowest at the most symmetric surfaces, we now estimate the measure at this point in $\mathcal{V}_{0,N}$ for large N. The estimates we find are consistent with some radius of convergence for the tachyon potential if the integrals are dominated by these configurations.

It is possible to do a very simple estimate. To this end consider the z plane and place one puncture at infinity with $|z| \geq 1$ its unit disk. In this way its mapping radius is just one. All other punctures will be distributed uniformly inside the disk. Because of area constraint we can imagine that each puncture will then carry a little disk of radius r, with $N\pi r^2 \sim \pi$ fixing the radius to be $r \sim 1/\sqrt{N}$. The mapping radius of each of these disks will be r. Another of the disks will be fixed at 0, and another to $2r$. We can now estimate the measure, which is the

integrand in (4.19), where in dealing with the three special punctures we make use of (3.18). We have then

$$d\mu_{\rm sym} \sim \frac{(2r)^2}{r^2\cdot r^2\cdot 1}\prod_{I=1}^{N-3}\frac{{\rm d}x_{_I}{\rm d}y_{_I}}{r^2} \sim N^{N-2}\prod_{I=1}^{N-3}{\rm d}x_{_I}{\rm d}y_{_I}\,. \tag{5.12}$$

Since all the punctures, except for the one at infinity, are inside the unit disk, we can compare the measure given above with the measure in the planar configuration. In that case the measure coefficient went like N^{4N} and now it essentially goes like N^N, which is much smaller, as we expected.⋆ We can also repeat the estimate we did for the integration over moduli space for the planar configuration, and again, we just get an extra multiplicative factor of N, which is irrelevant. Therefore, if we assume this configuration dominates we find $\mathrm{v}_{_N} \sim N^N$ and the tachyon potential $\sum \frac{\mathrm{v}_N}{N!}\tau^N$ would have some radius of convergence.

5.4. Analytic Continuation and Divergences

Here we want to discuss what happens if we do string field theory with stubs of length l. It is well-known that as the length of the stubs goes to infinity $l \to \infty$ then the region of moduli space corresponding to $\mathcal{V}_{0,4}$ approaches the full $\mathcal{M}_{0,4}$. In this case the coefficient of the quartic term in the potential will go into the full off-shell amplitude for scattering of four tachyons at zero momentum. We will examine the measure of integration for the tachyon potential as we approach the boundaries of moduli space and see that we would get a divergence corresponding to a tachyon of zero momentum propagating over long times. We believe this divergence is unphysical, and that the correct approach is to define the amplitude by analytic continuation from a region in the parameter space of the external momenta where the amplitude converges. When the full off-shell amplitude is built from the vertex contribution and the Feynman diagram contribution, analytic continuation is necessary for the Feynman part.

We therefore examine the off-shell formula for the evaluation of the four string vertex for general off-shell tachyons. What we need is the expression given in Eqn.(4.14) integrated over $\mathcal{V}_{0,4}$

$$A_{p_1\cdots p_4} = \frac{2}{\pi}\int\limits_{\mathcal{V}_{0,4}} {\rm d}x\,{\rm d}y\,\frac{|z|^{2p_1p_2}|z-1|^{2p_1p_3}}{\rho_z^{2-p_1^2}\rho_0^{2-p_2^2}\rho_1^{2-p_3^2}\rho_\infty^{2-p_4^2}}\,, \tag{5.13}$$

where we have added subscripts to the mapping radii in order to indicate the position of the punctures. Let us now examine what happens as we attempt to integrate with $z \to 0$,

⋆ Notice that if the punctures in the planar configuration had remained in the boundary of the unit disk then the measure would have only diverged like N^{2N}. The conformal map that brought them all to the real line between 0 and 1 introduced an extra factor of N^{2N}. This suggests that the divergence may actually not be as strong as it seems at first sight.

corresponding to a degeneration where punctures one and two collide. In this region ρ_1 and ρ_∞ behave as constants, and we have that

$$A_{p_1\cdots p_4} \sim \int\limits_{|z|<c} \mathrm{d}x\,\mathrm{d}y\,\frac{|z|^{2p_1p_2}}{\rho_z^{2-p_1^2}\rho_0^{2-p_2^2}}\,, \tag{5.14}$$

As the puncture at z is getting close to the puncture at zero it is intuitively clear that the mapping radii $\rho_z \sim \rho_0 \sim |z|/2$ as these are the radii of the "largest" nonintersecting disks surrounding the punctures. Therefore

$$A_{p_1\cdots p_4} \sim \int\limits_{|z|<c} \frac{\mathrm{d}x\,\mathrm{d}y}{|z|^{4-p_1^2-p_2^2-2p_1p_2}} = \int\limits_{|z|<c} \frac{\mathrm{d}x\,\mathrm{d}y}{|z|^{4-(p_1+p_2)^2}}\,, \tag{5.15}$$

and we notice that the divergence is indeed controlled by the momentum in the intermediate channel. If all the momenta were set to zero before integration, we get a divergence of the form $\int \mathrm{d}r/r^3$. But the way to proceed is to do the integral in a momentum space region where we have no divergence

$$A_{p_1\cdots p_4} \sim \int\limits_{|z|<c} \frac{\mathrm{d}r}{r^{3-(p_1+p_2)^2}} \sim \frac{1}{[2-(p_1+p_2)^2]}\,, \tag{5.16}$$

and the final result does not show a divergence for $p_1 = p_2 = 0$. Notice also that the denominator in the result is nothing else than $L_0 + \bar{L}_0$ for the intermediate tachyon, if that tachyon were on shell, we would get a divergence due to it.

6. A Formula for General Off-Shell Amplitudes on the Sphere

In this section we will derive a general formula for off-shell amplitudes on the sphere. In string field theory those amplitudes are defined as integrals over subspaces $\mathcal{A}_N$ of the moduli space $\widehat{\mathcal{P}}_{0,N}$ of N-punctured spheres with local coordinates, up to phases, at the punctures. We will assume a real parameterization of $\mathcal{A}_N$ and derive an operator expression which expresses the amplitude as a multiple integral over these parameters. The new point here is the explicit description of the relevant antighost insertions necessary to obtain the integrand, and the discussion explaining why the result does not depend neither on the parameterization of the subspace nor on the choice of a global uniformizer on the sphere.

We also give a formula for the case when the space $\mathcal{A}_N$ is parametrized by complex coordinates. For this case we will emphasize the analogy between the **b**–insertions for moduli and the **b**–insertions necessary to have PSL$(2,\mathbb{C})$ invariance. Finally, we will show how the general formula works by re-deriving the off–shell Koba-Nielsen amplitude considered earlier.

6.1. An Operator Formula for N–String Forms on the Sphere

Recall that the state space $\widehat{\mathcal{H}}$ of closed string theory consists of the states in the conformal theory that are annihilated both by $L_0 - \bar{L}_0$ and by $b_0 - \bar{b}_0$. Following [18] we now assign an N–linear function on $\widehat{\mathcal{H}}$ to any subspace $\mathcal{A}$ of $\widehat{\mathcal{P}}_{0,N}$. The multilinear function is defined as an integral over $\mathcal{A}$ of a canonical differential form

$$\{\Psi_1, \Psi_2, \ldots, \Psi_N\}_{\mathcal{A}} = \int_{\mathcal{A}} \Omega^{(\dim \mathcal{A} - \dim \mathcal{M}_{0,N})0,N}_{\Psi_1 \Psi_2 \ldots \Psi_N} \,. \tag{6.1}$$

One constructs the forms by verifying that suitable forms in $\mathcal{P}_{0,N}$ do lead to well defined forms in $\widehat{\mathcal{P}}_{0,N}$. This is the origin of the restriction of the CFT state space to $\widehat{\mathcal{H}}$. The canonical $2(N-3)+k$–form $\Omega^{(k)0,N}$ on $\widehat{\mathcal{P}}_{0,N}$ is defined by its action on $2(N-3)+k$ tangent vectors $V_I \in T_{\Sigma_P}\widehat{\mathcal{P}}_{0,N}$ as

$$\Omega^{(k)0,N}_{\Psi_1,\Psi_2,\ldots\Psi_N}(V_1, \ldots, V_{2(N-3)+k}) = \left(\frac{i}{2\pi}\right)^{N-3} \langle \Sigma_P | \mathbf{b}(\mathbf{v}_1) \cdots \mathbf{b}(\mathbf{v}_{2(N-3)+k}) | \Psi^N \rangle \,. \tag{6.2}$$

Here the surface state $\langle \Sigma_P |$ is a bra living in $(\mathcal{H}^*)^{\otimes N}$ and represents the punctured Riemann surface Σ_P. The symbol $\mathbf{v}_I$ denotes a Schiffer variation representing the tangent V_I, and

$$\mathbf{b}(\mathbf{v}) = \sum_{I=1}^{N} \oint_{w_I=0} \frac{dw_I}{2\pi i} v^{(I)}(w_I) b^{(I)}(w_I) + \oint_{\bar{w}_I=0} \frac{d\bar{w}_I}{2\pi i} \bar{v}^{(I)}(\bar{w}_I) \bar{b}^{(I)}(\bar{w}_I) \,. \tag{6.3}$$

Recall that a Schiffer variation for an N-punctured surface (in our present case a sphere) is an N–tuple of vector fields $\mathbf{v} = \left(v^{(1)}, v^{(2)}, \cdots, v^{(N)}\right)$ where the vector $v^{(k)}$ is a vector field defined in the coordinate patch around the k–th puncture.⋆ Let w_k be the local coordinate around the k–th puncture P_k ($w_k(P_k) = 0$). The variation defines a new N–punctured Riemann sphere with a new chosen local coordinates $w'_k = w_k + \epsilon v^{(k)}(w_k)$. The new k-th puncture is defined to be at the point P'_k such that $w'_k(P'_k) = 0$. It follows that the k–th puncture is shifted by $-\epsilon v^{(k)}(P_k)$. For any tangent $V \in T_P\widehat{\mathcal{P}}_{0,N}$ there is a corresponding Schiffer vector. Schiffer vectors are unique up to the addition of vectors that arise from the restriction of holomorphic vectors on the surface minus the punctures.

⋆ In general the vector fields $v^{(k)}$ are defined on some annuli around the punctures and do not extend holomorphically to the whole coordinate disk, in order to represent the change of modulus of the underlying non-punctured surface (see [18]). For $g = 0$ the underlying surface is the Riemann sphere and has no moduli. Therefore, the Schiffer vectors can be chosen to be extend throughout the coordinate disk.

Note that the insertions $\mathbf{b}(\mathbf{v})$ are invariantly defined, they only depend on the Schiffer vector, and do not depend on the local coordinates. Indeed b is a primary field of conformal dimension 2 or a holomorphic 2–tensor. Being multiplied by a holomorphic vector field v it produces a holomorphic 1–form, whose integral $\oint_{w=0} \frac{\mathrm{d}w}{2\pi i} v(w) b(w)$ is well-defined and independent of the contour of integration.

In order to evaluate (6.1) we choose some real coordinates $\lambda_1, \ldots, \lambda_{\dim \mathcal{A}}$ on $\mathcal{A}$. Let $\{V_{\lambda_k}\} = \partial/\partial\lambda_k$ be the corresponding tangent vectors, and let $\{\mathrm{d}\lambda_k\}$ be the corresponding dual one-forms, *i.e.* $\mathrm{d}\lambda_k(V_{\lambda_l}) = \delta_{k,l}$. Using $\{V_{\lambda_k}\}$ we can rewrite (6.1) as

$$\{\Psi_1, \cdots, \Psi_N\}_{\mathcal{A}} = \left(\frac{i}{2\pi}\right)^{N-3} \int \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_{\dim \mathcal{A}} \langle \Sigma_P | \mathbf{b}(\mathbf{v}_{\lambda_1}) \cdots \mathbf{b}(\mathbf{v}_{\lambda_{\dim \mathcal{A}}}) |\Psi_1\rangle \cdots |\Psi_N\rangle \,. \tag{6.4}$$

In order to continue we must parameterize $\mathcal{A}$ as it sits in the moduli space $\widehat{\mathcal{P}}_{0,N}$. Let w_I be a local coordinate around the I–th puncture. Given a global uniformizer z on the Riemann sphere we can represent w_I by as an invertible analytic function $w_I(z)$ defined on some disk in the z–plane which maps the disk to a standard unit disk $|w_I| < 1$. The inverse map $h_I = w_I^{-1}$ is therefore an analytic function on a unit disk. Therefore, N functions $h_I(w_I)$ define a point in $\widehat{\mathcal{P}}_{0,N}$, the sphere with N punctures at $h_I(0)$ and local coordinates given by $w_I(z) = h_I^{-1}(z)$. The embedding of $\mathcal{A}$ in $\widehat{\mathcal{P}}_{0,N}$ is then represented by a set of N holomorphic functions parameterized by the real coordinates λ_k on $\mathcal{A}$: $\{h_1(\{\lambda_k\}; w_1), \cdots, h_N(\{\lambda_k\}; w_N)\}$. It is well known how to write the state $\langle \Sigma_P|$ in terms of h_I's (see [24]).

$$\begin{aligned} \langle \Sigma_P | = 2 \cdot \int \prod_{I=1}^{N} \mathrm{d}p_I \, (2\pi)^D \delta^D \left(\sum p_I\right) \bigotimes_{I=1}^{N} \langle \mathbf{1}^c, p_I | \int \mathrm{d}^2\zeta^1 \mathrm{d}^2\zeta^2 \mathrm{d}^2\zeta^3 \\ \cdot \exp\left(E(\alpha) + F(b,c) - \sum_{n=1}^{3} \sum_{m \geq -1} \left(\zeta^n M_m^{nJ} b_m^{(J)} - \bar{\zeta}^n \overline{M_m^{nJ}} \bar{b}_m^{(J)} \right) \right). \end{aligned} \tag{6.5}$$

where repeated uppercase indices $I, J \cdots$, are summed over the N values they take. In here

$$\begin{aligned} E(\alpha) &= -\frac{1}{2} \sum_{n,m \geq 0} \left(\alpha_n^{(I)} N_{nm}^{IJ} \alpha_m^{(J)} + \bar{\alpha}_n^{(I)} \overline{N_{nm}^{IJ}} \bar{\alpha}_m^{(J)} \right) , \\ F(b,c) &= \sum_{\substack{n \geq 2 \\ m \geq -1}} \left(c_n^{(I)} \tilde{N}_{nm}^{IJ} b_m^{(J)} + \bar{c}_n^{(I)} \overline{\tilde{N}_{nm}^{IJ}} \bar{b}_m^{(J)} \right) . \end{aligned} \tag{6.6}$$

The vacua satisfy $\langle \mathbf{1}^c, p | \mathbf{1}, q\rangle = \delta(p - q)$, and the odd Grassmann variables ζ^n are integrated using

$$\int \mathrm{d}^2\zeta^1 \mathrm{d}^2\zeta^2 \mathrm{d}^2\zeta^3 \;\; \zeta^1\bar{\zeta}^1 \; \zeta^2\bar{\zeta}^2 \; \zeta^3\bar{\zeta}^3 \equiv 1 \,. \tag{6.7}$$

Note that the effect of this integration is to give the product of six antighost insertions coming from the last term in the exponential in Eqn.(6.5). The minus sign in front of this term is

actually irrelevant (as the reader can check) but it was included for later convenience. A bar over a number means complex conjugate while a bar over an operator is used in order to distinguish the left–moving modes from right–moving ones. The *Neumann coefficients* N^{IJ}_{mn} and $\tilde{N}^{IJ}_{mn}$ are given by the following formulae:

$$
\begin{aligned}
N^{IJ}_{00} &= \begin{cases} \log(h'_I(0)), & I = J \\ \log(h_I(0) - h_J(0)), & I \neq J \end{cases} \\
N^{IJ}_{0n} &= \frac{1}{n} \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{-n} h'_J(w) \frac{-1}{h_I(0) - h_J(w)}\,, \\
N^{IJ}_{mn} &= \frac{1}{m} \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} z^{-m} h'_I(z) \frac{1}{n} \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{-n} h'_J(w) \frac{1}{(h_I(z) - h_J(w))^2}\,, \\
\tilde{N}^{IJ}_{mn} &= \frac{1}{m} \oint_{z=0} \frac{\mathrm{d}z}{2\pi i} z^{-m+1} (h'_I(z))^2 \frac{1}{n} \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{-n-2} (h'_J(w))^{-1} \frac{1}{h_I(z) - h_J(w)}\,.
\end{aligned}
\tag{6.8}
$$

Moreover,

$$
M^{nJ}_m = \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{-m-2} (h'_J(w))^{-1} [h_J(w)]^{n-1}, \quad n = 1, 2, 3\,. \tag{6.9}
$$

<u>Antighost Insertions.</u> Now let us show how to take the **b** insertions into account. In order to calculate the **b** insertion associated to a tangent vector $V \in T_\Sigma \widehat{\mathcal{P}}_{0,N}$, we must find the Schiffer vector (field) that realizes the deformation of the surface Σ specified by V. Consider a line $c(t)$ in $\widehat{\mathcal{P}}_{0,N}$ parameterized by the real parameter t: $c : [0,1] \to \widehat{\mathcal{P}}_{0,N}$, such that $\Sigma = c(0)$, and the tangent vector to the curve is $V = c_* \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$. We will see now how one can use this setup to define in a natural way a vector field on the neighborhood of the punctures of the Riemann surface Σ. This vector field is the Schiffer vector.

We can represent the curve $c(t)$ by N functions $h_I(t; w)$, holomorphic in w, and parameterized by t. Choose a fixed value w_0 of the w disk. We now define a map $f^{w_0} : c(t) \to \Sigma$ from the curve $c(t)$ to a curve on the surface Σ. The function f^{w_0} takes $c(t)$ to $z_I(t) = h_I(t, w_0)$ for each value of t. We can now use the map f^{w_0} to produce a push-forward map of vectors $f^{w_0}_* : Tc \to T_{h_I(0,w_0)}\Sigma$. In this way we can produce the vector $f^{w_0}_* V \in T_{z_I(0,w_0)}\Sigma$. By varying the value of w_0 we obtain a vector *field* on the neighborhood of the I-th puncture. We claim that this vector field, with a minus sign, is the Schiffer vector. In components, and with an extra minus sign, the pullback gives

$$
v^{(I)}(z) = -\frac{\partial h_I}{\partial t}\,(t; w_I(z))\,, \tag{6.10}
$$

It is useful to refer the Schiffer vector to the local coordinate w_I. We then find, by pushing

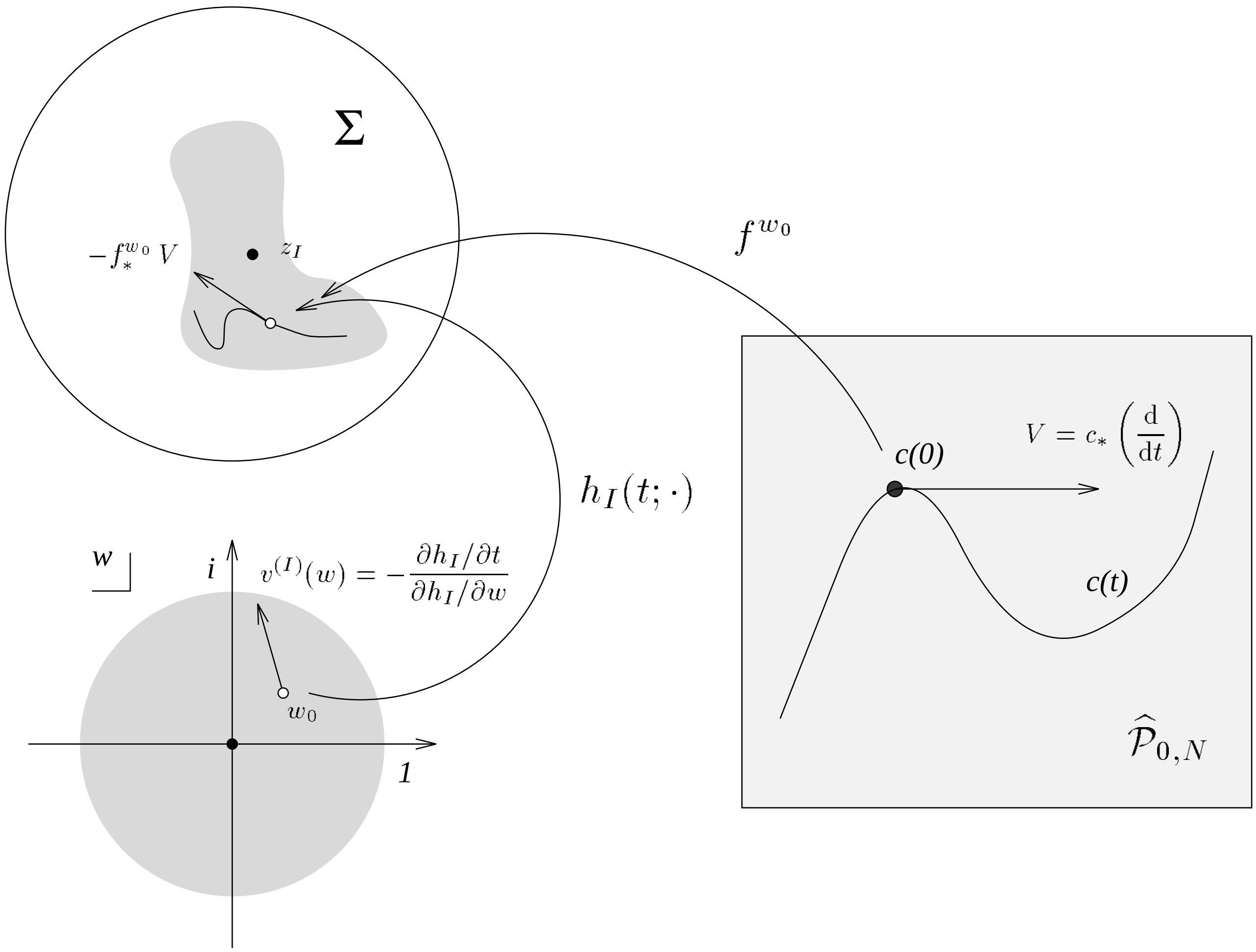

Figure 2. We show how to obtain a Schiffer vector field associated to a tangent vector in $\widehat{\mathcal{P}}_{0,N}$. Shown are the Riemann surface Σ and the local coordinate plane w.

the vector further

$$v^{(I)}(w_I) = -\left(\frac{\partial h_I}{\partial w_I}\right)^{-1} \cdot \frac{\partial h_I(t; w_I)}{\partial t}. \tag{6.11}$$

By definition, the Schiffer vector $\mathbf{v}(V)$ corresponding to the vector V is given by the collection of vector fields $\mathbf{v} = \Big(v^{(1)}(w_1), \cdots, v^{(N)}(w_N)\Big)$. If we define the vector $V_{\lambda_k} \in T\widehat{\mathcal{P}}_{0,N}$ to be the tangent associated to the coordinate curve parameterized by λ_k, we then write the Schiffer variation for V_{λ_k} as:

$$v^{(I)}_{\lambda_k}(w_I) = -\frac{1}{h'_I(w_I)} \frac{\partial h_I(\lambda; w_I)}{\partial \lambda_k}\,, \tag{6.12}$$

where $h'_I(w_I) \equiv (\partial h_I/\partial w_I)$.

Before proceeding any further, let us confirm that the 'natural' vector we have obtained is indeed the Schiffer vector. This is easily done. Let p denote a point in the Riemann surface Σ and let $w_{_I}(p)$ denote its local coordinate. By definition, the Schiffer vector defines a new coordinate $w'_{_I}(p)$ as $w'_{_I}(p) = w(p) + dt\, v^I(w(p))$, where t is again a parameter for the deformation. Since the z-coordinate of the point p does not change under the deformation, we must have that $h_{_I}(t+dt, w'_{_I}(p)) = h_{_I}(t, w_{_I}(p))$. Upon expansion of this last relation one immediately recovers Eqn.(6.11).

One more comment is in order. What happened to the usual ambiguity in choosing Schiffer vectors? Schiffer vectors are ambiguous since there are nonvanishing N-tuples that do not induce any deformation. This happens when the N-tuples can be used to define a holomorphic vector on the surface minus the punctures. In our case the ambiguity is due to the fact that the functions $h_{_I}(\lambda, w)$ can be composed with any PSL(2,$\mathbb{C}$) transformation S in the form $S \circ h_{_I}(\lambda, w)$. We will come back to this point later.

Using Eqn.(6.3) we can now write $\mathbf{b}(\mathbf{v}^\lambda)$ as

$$\mathbf{b}(\mathbf{v}_{\lambda_k}) = -\sum_{m\geq -1} \left(B_m^{kJ} b_m^{(J)} + \overline{B_m^{kJ}} \bar{b}_m^{(J)} \right), \tag{6.13}$$

where, as usual, the repeated index J is summed over the number of punctures, and

$$B_m^{kJ} = \oint\limits_{w=0} \frac{\mathrm{d}w}{2\pi i}\, w^{-m-2}\, \frac{1}{h'_{_J}(w)}\, \frac{\partial h_{_J}(\lambda, w)}{\partial \lambda_k}\,. \tag{6.14}$$

The range $m \geq -1$ has been obtained because the Schiffer vectors can be chosen to be holomorphic and not to have poles at the punctures (this will not be the case for higher genus surfaces).

Let us now treat the b insertions in a way similar to that used for the zero modes in (6.5). Let $\zeta_{_I}$ and $\eta_{_I}$ be anti-commuting variables, then

$$\int \mathrm{d}\xi_1 \ldots \mathrm{d}\xi_n \mathrm{e}^{\xi_1\eta_1+\cdots+\xi_n\eta_n} = \int \mathrm{d}\xi_1 \mathrm{e}^{\xi_1\eta_1} \cdots \int \mathrm{d}\xi_n \mathrm{e}^{\xi_n\eta_n} = \eta_1 \cdots \eta_n. \tag{6.15}$$

This observation allows us to represent the product of $\mathbf{b}$ insertions in (6.2) as an integral of an exponent.

$$\mathbf{b}(\mathbf{v}_{\lambda_1}) \cdots \mathbf{b}(\mathbf{v}_{\lambda_{\dim \mathcal{A}}}) = \int \prod_{k=1}^{\dim \mathcal{A}} \mathrm{d}\xi^k \exp\Big(-\sum_{k=1}^{\dim \mathcal{A}} \sum_{m\geq -1} \xi^k \left(B_m^{kJ} b_m^{(J)} + \overline{B_m^{kJ}} \bar{b}_m^{(J)} \right) \Big), \tag{6.16}$$

where the ξ^n's are real Grassmann odd variables. The multi-linear product (6.1) now assumes

the form

$$
\begin{aligned}
\{\Psi^N\}_{\mathcal{A}} = 2\left(\frac{i}{2\pi}\right)^{N-3} \int \prod_{I=1}^{N} \mathrm{d}^D p_{_I} (2\pi)^D \delta^D\left(\sum_{I=1}^{N} p_{_I}\right) \bigotimes_{I=1}^{N} \langle \mathbf{1}^c, p_{_I}| \int \prod_{k=1}^{\dim \mathcal{A}} \mathrm{d}\lambda_k \prod_{k=1}^{\dim \mathcal{A}} \mathrm{d}\xi^k \\
\int \mathrm{d}^2\zeta^1 \mathrm{d}^2\zeta^2 \mathrm{d}^2\zeta^3 \exp\Bigg(E(\alpha) + F(b,c) - \sum_{k=1}^{3} \sum_{m\geq -1} \left(\zeta^k M_m^{kJ} b_m^{(J)} - \bar{\zeta}^k \overline{M_m^{kJ}} \bar{b}_m^{(J)}\right) \\
- \sum_{k=1}^{\dim \mathcal{A}} \sum_{m\geq -1} \xi^k \left(B_m^{kJ} b_m^{(J)} + \overline{B_m^{kJ}} \bar{b}_m^{(J)}\right) \Bigg) |\Psi^N\rangle,
\end{aligned}
\tag{6.17}
$$

The above formula together with (6.6),(6.8), (6.9) and (6.14) gives a closed expression for a multi-linear form associated with $\mathcal{A} \subset \widehat{\mathcal{P}}_{0,N}$.

The resemblance of the last two terms appearing in the exponential is not a coincidence. While the last term arose from the antighost insertions for moduli, the first term, appearing already in the description of the surface state $\langle\Sigma|$, can be thought as the antighost insertions due to the Schiffer vectors that represent PSL(2,$\mathbb{C}$) transformations. This is readily verified. Consider the sphere with uniformizer z. The six globally defined vector fields are given by $v_k(z) = z^k$, and $v'_k(z) = iz^k$ with $k = 0, 1, 2$. Referring them to the local coordinates one sees that $v_k^I(w) = [h_{_I}(w)]^k / h'_{_I}(w)$ and $v'^I_k(w) = i[h_{_I}(w)]^k / h'_{_I}(w)$. As a consequence

$$
\mathbf{b}(v_k) = \sum_{m\geq -1} \left(M_m^{kJ}\, b_m + \overline{M_m^{kJ}}\, \bar{b}_m\right), \tag{6.18}
$$

$$
\mathbf{b}(v'_k) = i \sum_{m\geq -1} \left(M_m^{kJ}\, b_m - \overline{M_m^{kJ}}\, \bar{b}_m\right), \tag{6.19}
$$

where the M coefficients were defined in Eqn.(6.9). It is clear that the product of the six insertions precisely reproduces the effect of the first sum in the exponential of Eqn.(6.17).

In order to be used in (6.17) the subspace $\mathcal{A}$ is parametrized by some coordinates λ_k. The expression for the multilinear form is independent of the choice of coordinates; it is a well-defined form on $\mathcal{A}$.* Once the parametrization is chosen, the space $\mathcal{A}$ has to be described by the N functions $h_{_I}(\lambda, w)$. These functions, as we move on $\mathcal{A}$ are defined up to a *local* linear fractional transformation. At every point in moduli space we are free to change the uniformizer. Let us see why (6.17) has local PSL(2,$\mathbb{C}$) invariance. The bosonic Neumann coefficients N_{nm}^{IJ} are PSL(2,$\mathbb{C}$) invariant by themselves for $n, m \geq 1$. The sums

$\star$ This is easily verified explicitly. Under coordinate transformations, the product $\prod \mathrm{d}\lambda_i$ transforms with a Jacobian, and the product of antighost insertions, as a consequence of (6.14) transforms with the inverse Jacobian.

$\sum_{IJ} \alpha_0^{(I)} N_{00}^{IJ} \alpha_0^{(J)} = \sum_{IJ} N_{00}^{IJ} p_{_I} p_{_J}$ and $\sum_J N_{m0}^{IJ} \alpha_0^{(J)} = \sum_J N_{m0}^{IJ} p_{_J}$ can be shown to be invariant due to momentum conservation $\sum p_{_J} = 0$. A detailed analysis of PSL(2, $\mathbb{C}$) properties of the ghost part of a surface state have been presented by LeClair *et al.* in [24] for open strings. Their arguments can be readily generalized to our case. The only truly new part that appears in (6.17) is the last term in the exponential.

Under global PSL(2, $\mathbb{C}$) transformations, namely transformations of the form $h \to (ah + b)/(ch + d)$, with a, b, c and d independent of λ, this term is invariant because so is every coefficient B_m^{kJ}. Since a general local PSL(2, $\mathbb{C}$) tranformation can be written locally as a global transformation plus an infinitesimal local one, we must now show invariance under infinitesimal local transformations. These are transformations of the form

$$\widetilde{h}_{_I} = h_{_I} + a(\lambda) + b(\lambda) h_{_I} + c(\lambda) h_{_I}^2 \,, \tag{6.20}$$

for a, b, and c small. A short calculation shows that

$$-\frac{1}{\widetilde{h}'_{_J}} \frac{\partial \widetilde{h}_{_I}}{\partial \lambda_k} = -\frac{1}{h'_{_J}} \frac{\partial h_{_I}}{\partial \lambda_k} - \frac{1}{h'_{_J}} \Big[\frac{\partial a}{\partial \lambda_k} + \frac{\partial b}{\partial \lambda_k} h_{_I} + \frac{\partial c}{\partial \lambda_k} h_{_I}^2 \Big] \,. \tag{6.21}$$

On the left hand side we have the new Schiffer vector, and the first term the right hand side is the old Schiffer vector. We see that they differ by a linear superposition of the Schiffer vectors v_k^I and v'^I_k introduced earlier in our discussion of PSL(2, $\mathbb{C}$) transformations (immediately above Eqn.(6.18).) It follows that the extra contributions they make to the antighost insertions vanish when included in the multilinear form because the multilinear form already includes the antighost insertions corresponding to the Schiffer vectors generating PSL(2, $\mathbb{C}$).

<u>Complex Coordinates</u> In some applications the subspace $\mathcal{A}$ has even dimension and can be equipped with complex coordinates. Let $\dim^c \mathcal{A}$ denote the complex dimension of $\mathcal{A}$ and let $\{\lambda_1, \cdots, \lambda_{\dim^c \mathcal{A}}\}$ be a set of complex coordinates. The subspace $\mathcal{A}$ can now be represented by the collection of functions $\{h_{_I}(\{\lambda_k\}, \{\bar{\lambda}_k\}; w_{_I})\}$ with $I = 1, \cdots, N$. In order to derive a formula for this case we simply take the earlier result for two real insertions and pass to complex coordinates. We thus consider

$$\Omega^2 = d\lambda_1 \wedge d\lambda_2 \; d\xi^1 d\xi^2 \; \exp\Big[-\sum_{k=1}^{2} \xi^k \left(B_m^{kJ} b_m^{(J)} + \overline{B_m^{kJ}} \bar{b}_m^{(J)} \right) \Big] \,, \tag{6.22}$$

Using complex coordinates $\lambda'_1 = \lambda_1 + i\lambda_2$ and $\xi'^1 = \xi^1 + i\xi^2$, and letting $\int d^2\xi' \, \xi' \bar{\xi}' \equiv 1$, we can write the above as

$$\Omega^2 = d\lambda'_1 \wedge d\bar{\lambda}'_1 \; d^2\xi'_1 \; \exp\Big[-\xi'^1 \left(B_m^{1J} b_m^{(J)} + \overline{B_m^{\bar{1}J}} \bar{b}_m^{(J)} \right) + \bar{\xi}'^1 \left(B_m^{\bar{1}J} b_m^{(J)} + \overline{B_m^{1J}} \bar{b}_m^{(J)} \right) \Big] \,, \tag{6.23}$$

where we have defined

$$
\begin{aligned}
B_m^{kJ} &= \oint\limits_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{-m-2} \frac{1}{h'_{_J}(w)} \frac{\partial h_{_J}(\lambda,\bar{\lambda};w)}{\partial \lambda_k}, \\
B_m^{\bar{k}J} &= \oint\limits_{w=0} \frac{\mathrm{d}w}{2\pi i} w^{-m-2} \frac{1}{h'_{_J}(w)} \frac{\partial h_{_J}(\lambda,\bar{\lambda};w)}{\partial \bar{\lambda}_k}.
\end{aligned}
\tag{6.24}
$$

Note that the $B_m^{\bar{k}J}$ coefficients do not vanish because the embedding of $\mathcal{A}$ in $\widehat{\mathcal{P}}_{0,N}$ need not be holomorphic and, as a consequence, the derivatives $\partial h_{_I}/\partial\bar{\lambda}_k$ need not vanish. Using this result we can now rewrite (6.17) for the case of complex coordinates for moduli

$$
\begin{aligned}
\{\Psi^N\}_{\mathcal{A}} = 2\left(\frac{i}{2\pi}\right)^{N-3} & \int \prod_{I=1}^{N} \mathrm{d}^D p_{_I} (2\pi)^D \delta^D \left(\sum p_{_I}\right) \bigotimes_{I=1}^{N} \langle \mathbf{1}^c, p_{_I}| \int\limits_{\mathcal{A}} \prod_{k=1}^{\dim^c \mathcal{A}} \mathrm{d}\lambda_k \wedge \mathrm{d}\bar{\lambda}_k \prod_{k=1}^{\dim^c \mathcal{A}} \mathrm{d}^2 \xi^k \\
& \prod_{k=1}^{3} \mathrm{d}^2 \zeta^k \, \exp\Bigg(E(\alpha) + F(b,c) - \sum_{k=1}^{3} \left(\zeta^k M_m^{kJ} b_m^{(J)} - \bar{\zeta}^k \overline{M_m^{kJ}} \bar{b}_m^{(J)} \right) \\
& - \sum_{k=1}^{\dim^c \mathcal{A}} \left[\xi^k \left(B_m^{kJ} b_m^{(J)} + \overline{B_m^{\bar{k}J}} \bar{b}_m^{(J)} \right) - \bar{\xi}^k \left(B_m^{\bar{k}J} b_m^{(J)} + \overline{B_m^{kJ}} \bar{b}_m^{(J)} \right) \right] \Bigg) |\Psi^N\rangle,
\end{aligned}
\tag{6.25}
$$

where $m \geq -1$ for the implicit oscillator sum. It is useful to bring out the similarity between the antighost insertions for $\mathrm{PSL}(2,\mathbb{C})$ and those for moduli. In order to achieve this goal we introduce $\xi^{\dim^c \mathcal{A}+k} \equiv \zeta^k$ for $k = 1,2,3$.

$$
\begin{aligned}
\{\Psi^N\}_{\mathcal{A}} = 2\left(\frac{i}{2\pi}\right)^{N-3} & \int \prod_{I=1}^{N} \mathrm{d}^D p_{_I} (2\pi)^D \delta^D \left(\sum p_{_I}\right) \bigotimes_{I=1}^{N} \langle \mathbf{1}^c, p_{_I}| \int\limits_{\mathcal{A}} \prod_{k=1}^{\dim^c \mathcal{A}} \mathrm{d}\lambda_k \wedge \mathrm{d}\bar{\lambda}_k \\
& \prod_{k=1}^{\dim^c \mathcal{A}+3} \mathrm{d}^2 \xi^k \cdot \exp\Bigg(E(\alpha) + F(b,c) - \sum_{k=1}^{\dim^c \mathcal{A}+3} \left(\xi^k \mathcal{B}_m^{kJ} b_m^{(J)} - \bar{\xi}^k \overline{\mathcal{B}_m^{kJ}} \bar{b}_m^{(J)} \right) \\
& - \sum_{k=1}^{\dim^c \mathcal{A}} \left(\xi^k \, \overline{B_m^{\bar{k}J}} \, \bar{b}_m^{(J)} - \bar{\xi}^k B_m^{\bar{k}J} b_m^{(J)} \right) \Bigg) |\Psi^N\rangle,
\end{aligned}
\tag{6.26}
$$

where the script style $\mathcal{B}$ matrix elements are defined as

$$
\mathcal{B}_m^{kJ} = \begin{cases} B_m^{kJ}, & \text{for } k \leq \dim^c \mathcal{A} \\ M_m^{(k-\dim^c \mathcal{A})J}, & \text{for } k - \dim^c \mathcal{A} = 1,2,3. \end{cases}
\tag{6.27}
$$

This concludes our construction of off-shell amplitudes as forms on moduli spaces of punctured spheres.

6.2. Application to Off-Shell Tachyons

Let us see how the formulae derived above work for the case of N tachyons with arbitrary momenta. This particular example allows us to confirm our earlier calculation of off-shell tachyon amplitudes. Specifically, we are going to evaluate the multilinear function $\{\tau_{p_1}, \cdots, \tau_{p_N}\}$ where $|\tau_{p_i}\rangle = c_1\bar{c}_1|\mathbf{1}, p_i\rangle$. In this case the state to be contracted with the bra representing the multilinear function is $|\tau^N\rangle = \otimes_{I=1}^{N} c_1^{(I)}\bar{c}_1^{(I)}|\mathbf{1}, p_I\rangle$. Upon contraction with this state we will only get contributions from $b_{-1}^{(I)}$, $\bar{b}_{-1}^{(I)}$, and the matter zero modes $\alpha_0^{(I)} = \bar{\alpha}_0^{(I)} = ip_I$.

We will use as moduli the complex coordinates $z_1, \cdots, z_{N-3}$ representing the position of the first $(N-3)$ punctures. Therefore, $\lambda_k = z_k$, for $k = 1, \ldots, N-3$, and we must use Eqn.(6.26) to calculate the multilinear function. Our setting of the z-coordinates as moduli implies that the functions h_J take the form

$$h_J(z, \bar{z}, w) = z_J + a(z, \bar{z})w + \cdots . \tag{6.28}$$

It then follows that

$$M_{-1}^{kJ} = \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} \frac{1}{w} (h'_J(w))^{-1} [h_J(w)]^{k-1} = \frac{z_J^{k-1}}{h'_J(0)} , \tag{6.29}$$

and furthermore

$$B_{-1}^{kJ} = \oint_{w=0} \frac{\mathrm{d}w}{2\pi i} \frac{1}{w} \frac{1}{h'_J(w)} \frac{\partial h_J}{\partial \lambda_k} = \frac{\delta^{kJ}}{h'_J(0)} , \tag{6.30}$$

while $B_m^{\bar{k}J} = 0$, since $\partial h_J / \partial \bar{z}_k = 0$. With this information, back in (6.26) we find

$$\begin{aligned}\{\tau_{p_1}, \cdots, \tau_{p_N}\}_{\mathcal{A}} = 2 \left(\frac{i}{2\pi}\right)^{N-3} (2\pi)^D \delta^D(\mathbf{0}) \int \prod_{k=1}^{N-3} \mathrm{d}z_k \wedge \mathrm{d}\bar{z}_k \prod_{k=1}^{N} \mathrm{d}^2\xi^k \\ \cdot \langle \mathbf{1}^c | \exp \left(E(\alpha) - \sum_{k=1}^{N} \left(\xi^k \mathcal{B}_{-1}^{kJ} b_{-1}^{(J)} - \bar{\xi}^k \overline{\mathcal{B}_{-1}^{kJ}} \bar{b}_{-1}^{(J)} \right) \right) \prod_{I=1}^{N} c_1^{(I)} \bar{c}_1^{(I)} |\mathbf{1}\rangle .\end{aligned} \tag{6.31}$$

We can now calculate the bosonic contribution from $E(\alpha)$

$$\begin{aligned}\exp(E(\alpha)) &= \exp\Big(-\frac{1}{2} \sum_{I,J=1}^{N} \Big(\alpha_0^{(I)} N_{00}^{IJ} \alpha_0^{(J)} + \bar{\alpha}_0^{(I)} \overline{N_{00}^{IJ}} \bar{\alpha}_0^{(J)} \Big) \Big) \\ &= \exp\Big(-\frac{1}{2} \sum_{I,J=1}^{N} \Big(N_{00}^{IJ} + \overline{N_{00}^{IJ}} \Big) p_I p_J \Big) \\ &= \prod_{I<J} \left(\frac{|h_I(0) - h_J(0)|^2}{|h'_I(0)| \cdot |h'_J(0)|} \right)^{p_I p_J} = \prod_{I<J} \chi_{IJ}^{p_I p_J} ,\end{aligned} \tag{6.32}$$

where we have used the expression for Neumann coefficients (6.8), momentum conservation, and (3.7). We thus obtain

$$\{\tau_{p_1}, \cdots, \tau_{p_N}\}_{\mathcal{A}} = \frac{2}{\pi^{N-3}} (2\pi)^D \delta^D(\mathbf{0}) \int \prod_{k=1}^{N-3} \mathrm{d}x_k \, \mathrm{d}y_k \prod_{I<J} \chi_{IJ}^{p_I p_J}$$
$$\cdot \prod_{k=1}^{N} \mathrm{d}^2\xi^k \langle \mathbf{1}^c | \exp\Big(- \sum_{k=1}^{N} \Big(\xi^k \mathcal{B}_{-1}^{kJ} b_{-1}^{(J)} - \bar{\xi}^k \overline{\mathcal{B}_{-1}^{kJ}} \bar{b}_{-1}^{(J)} \Big) \Big) \prod_{I=1}^{N} c_1^{(I)} \bar{c}_1^{(I)} | \mathbf{1} \rangle \, . \tag{6.33}$$

Consider the second line of the above equation. The effect of the ghost state is to select the term in the exponential proportional to the product of all antighosts. Since

$$\langle \mathbf{1}^c | \prod_{I=1}^{N} b_{-1}^{(I)} \bar{b}_{-1}^{(I)} \prod_{I=1}^{N} c_1^{(I)} \bar{c}_1^{(I)} | \mathbf{1} \rangle = (-)^N \, , \tag{6.34}$$

we can write the second line of (6.31) using a second set of Grasmmann variables η^k

$$(-)^N \int \prod_{k=1}^{N} \mathrm{d}^2\xi^k \, \mathrm{d}^2\eta^k \exp\Big[\sum_{k,p=1}^{N} \Big(-\, \xi^k \, \mathcal{B}_{-1}^{kp} \, \eta^p + \bar{\xi}^k \, \overline{\mathcal{B}_{-1}^{kp}} \, \bar{\eta}^p \Big) \Big] \; = \; (-)^N \, \det(\mathcal{B}) \, \det(\bar{\mathcal{B}}) \, , \tag{6.35}$$

where in the last step we used a standard formula in Grassmann integration. We can now use Eqns. (6.29), (6.30), and (6.27) to calculate $|\det \mathcal{B}\,|^2$. We find

$$\begin{aligned} |\det \mathcal{B}|^2 &= |(z_{N} - z_{N-2})(z_{N} - z_{N-1})(z_{N-2} - z_{N-1})|^2 \prod_{I=1}^{N} \frac{1}{|h'_{I}(0)|^2} \, , \\ &= \chi^2_{N-2,N-1,N} \prod_{I=1}^{N-3} \frac{1}{\rho_I^2} \, , \end{aligned} \tag{6.36}$$

where we made use of the definition of the mapping radius and of Eqn.(3.10). We can now assemble the final form of the tachyon multilinear function. Back in (6.33) we have

$$\{\tau_{p_1}, \cdots, \tau_{p_N}\}_{\mathcal{A}} = (-)^N \frac{2}{\pi^{N-3}} \int \prod_{I=1}^{N-3} \frac{\mathrm{d}x_I \mathrm{d}y_I}{\rho_I^2} \, \chi^2_{N-2,N-1,N} \prod_{I<J} \chi_{IJ}^{q_I q_J} \cdot (2\pi)^D \delta(\mathbf{0}) \, . \tag{6.37}$$

which agrees precisely with the off-shell Koba-Nielsen formula (4.13).

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APPENDIX A: The tachyon potential and string field redefinitions.

Here we wish to discuss whether it is possible to make a field redefinition of the string field tachyon such that the string action is brought to a form where one could rule out the existence of a local minimum. Even at the level of open string field theory this seems hard to achieve. The tachyon potential is of the form $V \sim -\tau^2 + g\tau^3$. The cubic term produces a local minimum with a nonzero vacuum expectation value for τ. We are not allowed to just redefine τ to absorb the cubic term in the quadratic one; this is a non-invertible field redefinition. Using the massive fields is no help since the transformations must preserve the kinetic terms, and therefore should be of the form $\tau \to \tau + f(\phi_i, \tau)$ and $\phi_i \to \phi_i + g(\phi_i, \tau)$, with f and g functions that must start quadratic in the fields. Such transformations cannot eliminate the cubic term in the tachyon potential.

Let us examine the question of field redefinitions in a more stringy way. Assume it is possible to write the string action as

$$S = \int \mathrm{d}x \left[\mathcal{L}(\nabla\tau, \phi_i) + \tau^2(1 + f(\phi_i)) \right], \tag{A.1}$$

namely, that one can separate out a term just depending on derivatives of the tachyon field, and all other fields, and a quadratic term for the tachyon potential. The term $f(\phi_i)$ was included to represent couplings to fields like dilatons or background metric. If the above were true we would expect no perturbatively stable minimum for the tachyon (the factor $(1+f(\phi_i))$ is expected to be nonvanishing). We will now argue that the string action *cannot* be put in the form described in Eqn.(A.1) by means of a string field redefinition. It is therefore not possible to rule out a local minimum by such simple means.*

If Eqn.(A.1) holds, a constant infinitesimal shift of the tachyon field $\tau \to \tau + \epsilon$ would have the effect of shifting the action as

$$S \to S + 2\epsilon \int \mathrm{d}x\, \tau(1 + f(\phi_i)) + \mathcal{O}(\epsilon^2). \tag{A.2}$$

We should be able to prove such "low-energy tachyon theorem" with string field theory. For this we must find the change in the string action as we shift the string field as follows

$$|\Psi\rangle_1 \to |\Psi\rangle_1 + \epsilon|T_0\rangle_1 + \epsilon\langle h^{(2)}_{23}|\Psi\rangle_2|\mathcal{S}_{13}\rangle + \cdots, \tag{A.3}$$

where $|T_0\rangle = c_1\bar{c}_1|\mathbf{1}\rangle$, the dots indicate quadratic and higher terms in the string field, $\langle h^{(2)}_{23}|$ is a symmetric bra, and $|\mathcal{S}_{13}\rangle$ is the sewing ket [2]. Indeed the transformation of the string field

$\star$ It is not clear to us whether this result is in contradiction with that of Ref.[13], where presumably the relevant action is the effective action obtained after integrating out classically the massive fields.

cannot be expected to be a simple shift along the zero-momentum tachyon, since the string field tachyon should differ from the tachyon appearing in (A.1). If we now vary the string action (2.4) we find

$$S \rightarrow S + \epsilon\langle\Psi|c_0^- Q|T_0\rangle + \frac{1}{2}\epsilon\left[\langle V_{123}^{(3)}|T_0\rangle_3 + \langle h_{12}^{(2)}|(Q_1+Q_2)\right]|\Psi\rangle_1|\Psi\rangle_2 + \cdots . \tag{A.4}$$

We must now see that by a suitable choice of $\langle h_{12}^{(2)}|$ the variation of the action takes the form required by (A.2). Indeed, the term $\epsilon\,\tau$ arises from the $\epsilon\langle\Psi|c_0^- Q|T_0\rangle$ term in (A.4) since $c_0^- Q|T_0\rangle$ can only couple to the tachyon field in Ψ. Assume the function $f(\phi_i)$ in (A.2) is zero, in that case there is no extra variation in the action, and we must require that

$$\langle V_{123}^{(3)}|T_0\rangle_3 + \langle h_{12}^{(2)}|(Q_1+Q_2) = 0\,. \tag{A.5}$$

This equation cannot have solutions; acting once more with (Q_1+Q_2) we find that (A.5) requires that $\langle V_{123}^{(3)}|Q_3|T_0\rangle_3 = 0$ which cannot hold (recall $Q|T_0\rangle \neq 0$). Even if $f(\phi_i)$ is not zero, we do not expect solutions to exist. In this case we still must have that (A.5) should be zero contracted with any arbitrary two states, except when one of them is a zero momentum tachyon. Again using states of the form $(Q_1+Q_2)|a\rangle_1|b\rangle_1$ where neither $|a\rangle$ nor $|b\rangle$ is a zero momentum tachyon, we see that again the equation cannot be satisfied. This shows that there is no simple "low-energy tachyon theorem" that rules out a local minimum.

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